

JUNE 1978

PPPL-1455

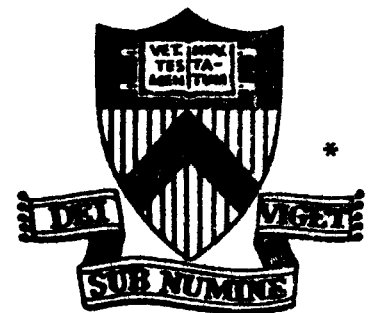
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THE COMPLEX MODIFIED KORTEWEG-
DEVRIES EQUATION, A NON-INTEGRABLE
EVOLUTION EQUATION

BY

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This work was supported by the U. S. Department of Energy
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The Complex Modified Korteweg-DeVries Equation,
a Non-Integrable Evolution Equation

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Abstract

The two-dimensional steady-state propagation of electrostatic waves is governed by $\partial v / \partial \tau + \partial^3 v / \partial \xi^3 + \partial(|v|^2 v) / \partial \xi = 0$, the Complex Modified Korteweg-DeVries equation. The properties of this equation are studied.

To be presented at the Symposium on Nonlinear (Soliton) Structure and Dynamics in Condensed Matter, Oxford, England, June 27-29, 1978.

THE COMPLEX MODIFIED KORTEWEG-DEVRIES EQUATION,
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1. Derivation of Equation

Some anisotropic media, e.g. a magnetized plasma, can support propagating electrostatic waves. We wish to study the dispersion and nonlinear self-modulation of such waves when the system is two-dimensional (x and y , x being the principal axis) and has reached a steady state [all field quantities $\sim \exp(-i\omega t)$, $\omega = \text{constant}$]. We will take the medium to be homogeneous and non-dissipative, and will assume that the dielectric tensor depends on the electric field amplitude squared. The equation for the complex electric potential, ϕ , is

$$\nabla \cdot \bar{K}(\nabla, |\nabla\phi|^2) \cdot \nabla\phi = 0 \quad (1)$$

We expand \bar{K} about $\nabla = 0$, $|\nabla\phi|^2 = 0$ (the long-wavelength, linear limit)

$$\bar{K}(\nabla, |\nabla\phi|^2) = \bar{K} + \epsilon(1/2)(\partial^2 \bar{K} / \partial \nabla \partial \nabla) : \nabla \nabla + \epsilon(\partial \bar{K} / \partial |\nabla\phi|^2) |\nabla\phi|^2, \quad (2)$$

where ϵ is a formal expansion parameter. Assuming $K_{yy} > 0 > K_{xx}$ then to order ϵ^0 (1) becomes the wave equation $-|K_{xx}| \partial^2 \phi / \partial x^2 + K_{yy} \partial^2 \phi / \partial y^2 = 0$.

Expanding ϕ about the right-going solution we let $\phi = \phi(\tau, \xi)$, where $\tau \propto y$ and $\xi \propto x - (|K_{xx}|/K_{yy})^{1/2} y$ and $\partial\phi/\partial\xi = O(1)$, $\partial\phi/\partial\tau = O(\epsilon)$. To order ϵ we obtain

$$v_\tau + v_{\xi\xi\xi} + (|v|^2 v)_\xi = 0, \quad (3)$$

where $v \propto \partial\phi/\partial\xi$, and subscripts denote differentiation. We call (3) the Complex Modified Korteweg-deVries equation [1].

2. Constants of the Motion

Only four constants of the motion are known. This is in contrast to equations soluble by the inverse scattering method which have an infinite number of constants of the motion. The four constants are:

$$I_1 = \int_{-\infty}^{\infty} v \, d\xi, \quad (4) \quad I_2 = \int_{-\infty}^{\infty} |v|^2 \, d\xi, \quad (5)$$

$$I_3 = \int_{-\infty}^{\infty} |v|^4 / 2 - |v_\xi|^2 \, d\xi, \quad (6) \quad I_4 = \int_{-\infty}^{\infty} |v|^2 / k \, dk. \quad (7)$$

In (7) k is the Fourier-transform variable conjugate to ξ . Three of these constants have physical interpretations. $l_1 = \text{const.}$ states that the electric field is derivable from a potential. $l_2 = \text{const.}$ and $l_4 = \text{const.}$ give the conservation of momentum and energy, i.e. the force and power balances.

3. Soluble Limits

Although (3) is not analytically soluble, it is closely related to the modified Korteweg-deVries equation,

$$v_\tau + v_{\xi\xi\xi} + \kappa |v|^2 v_\xi = 0, \quad (8)$$

which is soluble by the inverse scattering method [2]. To see this relation we rewrite (3) in two ways,

$$v_\tau + v_{\xi\xi\xi} + 3|v|^2 v_\xi = 2i|v|^2 v\theta_\xi, \quad (9)$$

$$v_\tau + v_{\xi\xi\xi} + |v|^2 v_\xi = -v|v^2|_\xi, \quad (10)$$

where $\theta = \arg(v)$. In the limits of slow and rapid phase variation, the righthand sides of (9) and (10) respectively are negligible and in these limits (3) reduces to (8), although the strength of the nonlinear term, κ , is different. When neither limit applies, we must solve (3) numerically.

4. Numerical Solution

We choose initial conditions of the form,

$$v(\tau=0, \xi) = A \text{sech}(\xi) \exp(ik_0 \xi). \quad (11)$$

Figs. 1 and 2 show two examples of the evolution. We see that there are two types of solitary pulses produced; one had a constant phase (Fig. 1), while the other is an envelope pulse (Fig. 2).

5. Constant Phase Pulses

The constant phase pulses are a special case of the solitons of (8). Their form is

$$v = \sqrt{2} a \text{sech}[a(\xi - \xi_0 - a^2\tau)] \exp(i\theta_0). \quad (12)$$

The area of these pulses is $\sqrt{2}\pi$. However, these pulses do not behave as solitons in (3). Fig. 3 shows the collision of two of these pulses which have different phases, θ_0 . We see that after the collision the phase and amplitude of the pulses have changed and some "radiation" is produced.

6. Envelope Pulses

The general form of the envelope pulses is $v(\tau, \xi) = V(\zeta) \exp(ik_0 \xi - i\omega_0 \tau)$, where V is complex, $\zeta = \xi - c\tau$, $c = a^2 - 3k_0^2$, and $\omega_0 = k_0(3a^2 - k_0^2)$. Here \underline{a} is the decay rate of V as $|\zeta| \rightarrow \infty$. V satisfies

$$(V_{\zeta\zeta} + |V|^2 V - a^2 V)_\zeta + ik_0(3V_{\zeta\zeta} + |V|^2 V - 3a^2 V) = 0. \quad (13)$$

For $k_0 = 0$, we recover the constant phase pulses. Numerically integrating (13), we find that for $0 < |k_0| \lesssim 0.5a$, V does not form a pulse, while for $|k_0| \gtrsim 0.5a$ we do obtain a pulse. The asymptotic form of the pulse for k_0 large is

$$V = \sqrt{6} a \operatorname{sech}(a\zeta) \{1 + i\epsilon \tanh(a\zeta) + \epsilon^2 [\tanh^2(a\zeta) - 1/2]/3 + O(\epsilon^3)\}, \quad (14)$$

where $\epsilon = a/k_0$. In the limit $\epsilon \rightarrow 0$ we recover the soliton of (8) (with $\kappa = 3$). The "absolute" area of V is (cf. the area of the constant phase pulses)

$$\int_{-\infty}^{\infty} |V| d\zeta = \sqrt{6} \pi [1 + \epsilon^2/4 + O(\epsilon^4)]. \quad (15)$$

7. Transition From Envelope to Constant Phase Pulses

Since solitary pulses do not exist for the full range of k_0/a it is clear that there cannot be a continuous transition from envelope to constant phase pulses. If we take an envelope pulse and alter the initial conditions in such a way that $|k_0| < 0.5a$, we see from (15) that it has sufficient area to break up into about 3 constant phase pulses. For initial conditions of the form of (11) this happens when $A \approx 2k_0$ (see Fig. 4). Fig. 5 shows schematically what pulses are produced for different initial conditions. The equivalent figure for (8) would consist of a single set of lines at $A(\kappa/6)^{1/2} = N - 1/2$.

Acknowledgments

We wish to thank J. D. Meiss and N. R. Pereira for making available their program for solving nonlinear evolution equations. One of us (FYFC) thanks the Theoretical Physics Group of Lawrence Berkeley Laboratory for its hospitality. This work was supported by USDoE (Contracts EY-76-C-02-3073 and EG-77-G-01-4107), NSF (Grant ENG-77-00340), and US Air Force (Grant AFOSR 77-3321).

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2. M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur: Stud. Appl. Math. 53, 249-315 (1974)

Figure Captions

Fig. 1 Evolution of (3) with initial conditions given by (11) with $A = 3, k_0 = 1$. The solid line, long dashes, and short dashes denote $|v|$, $\text{Re}(v)$, and $\text{Im}(v)$ respectively

Fig. 2 Same as Fig. 1, except $A = 5, k_0 = 3$

Fig. 3 Collision of two constant phase pulses

Fig. 4 Same as Fig. 2, except $A = 6$

Fig. 5 Schematic showing numbers and types of pulses produced with initial conditions given by (11)

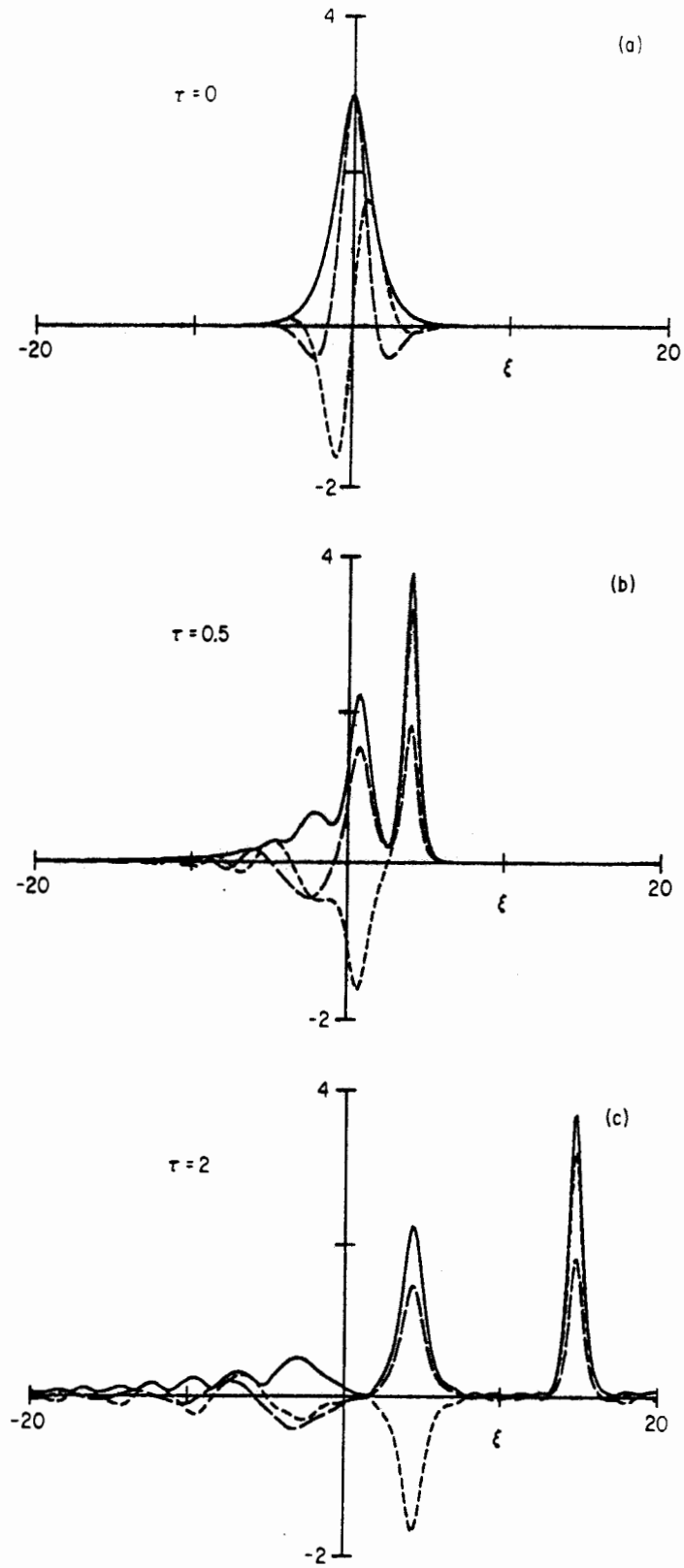


Fig. 1. 782165

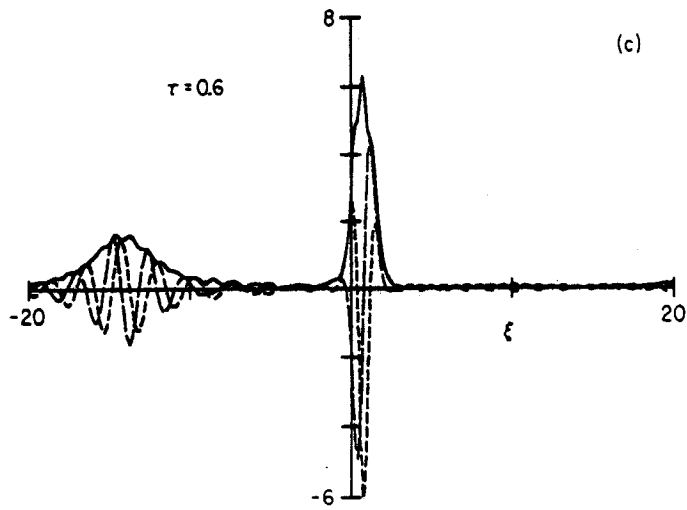
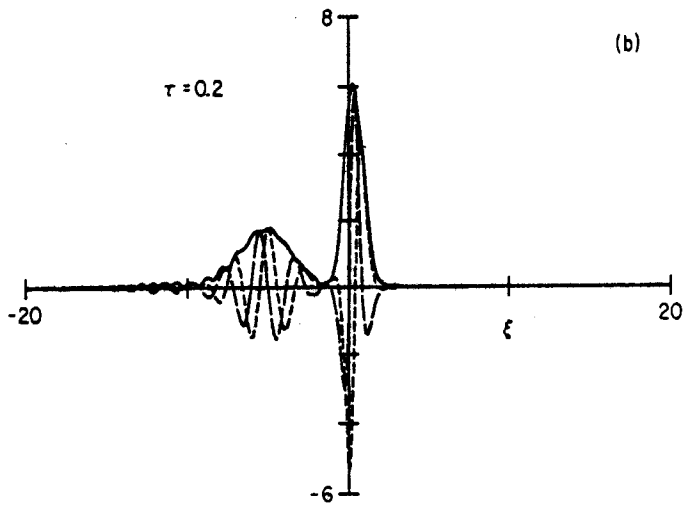
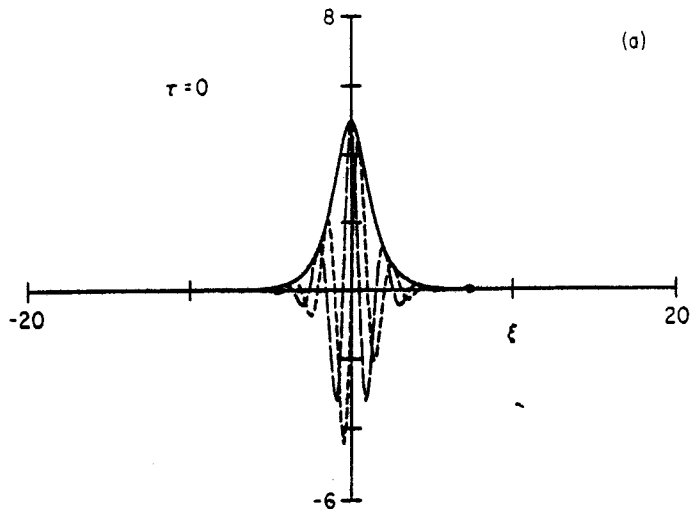


Fig. 2. 782166

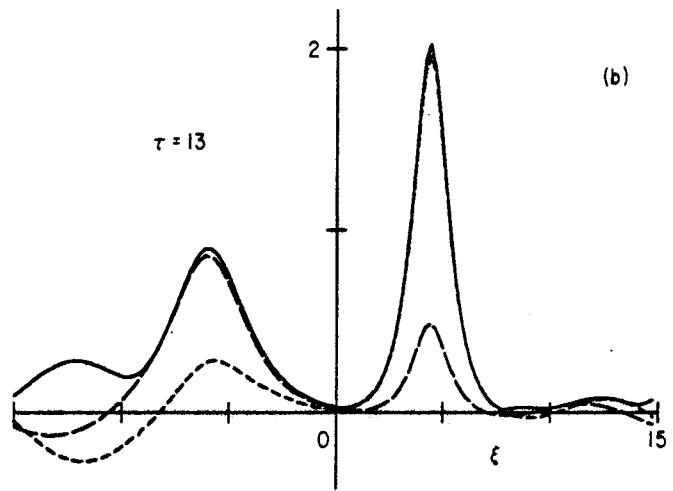
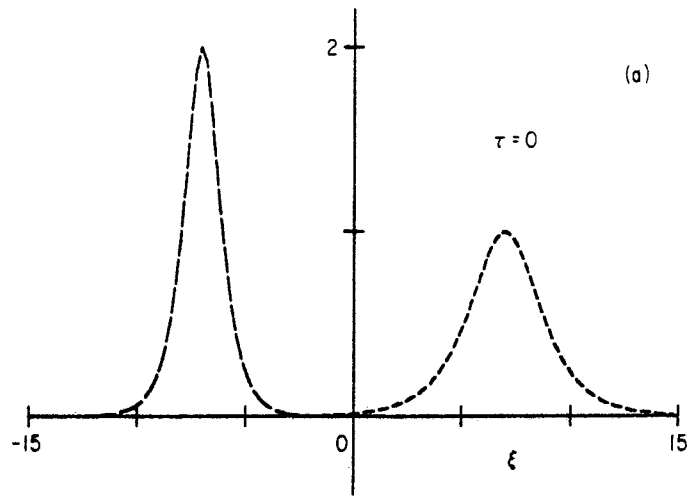


Fig. 3. 782164

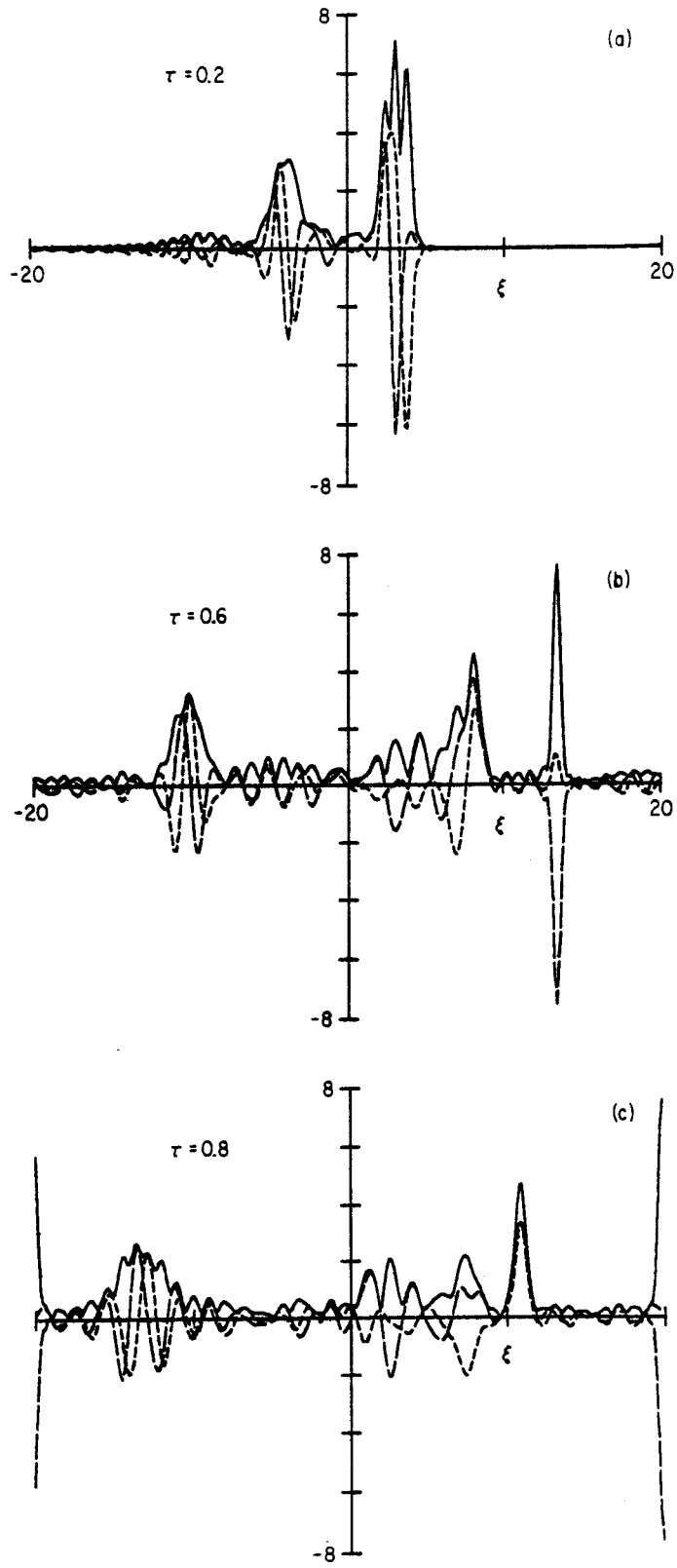


Fig. 4. 782163

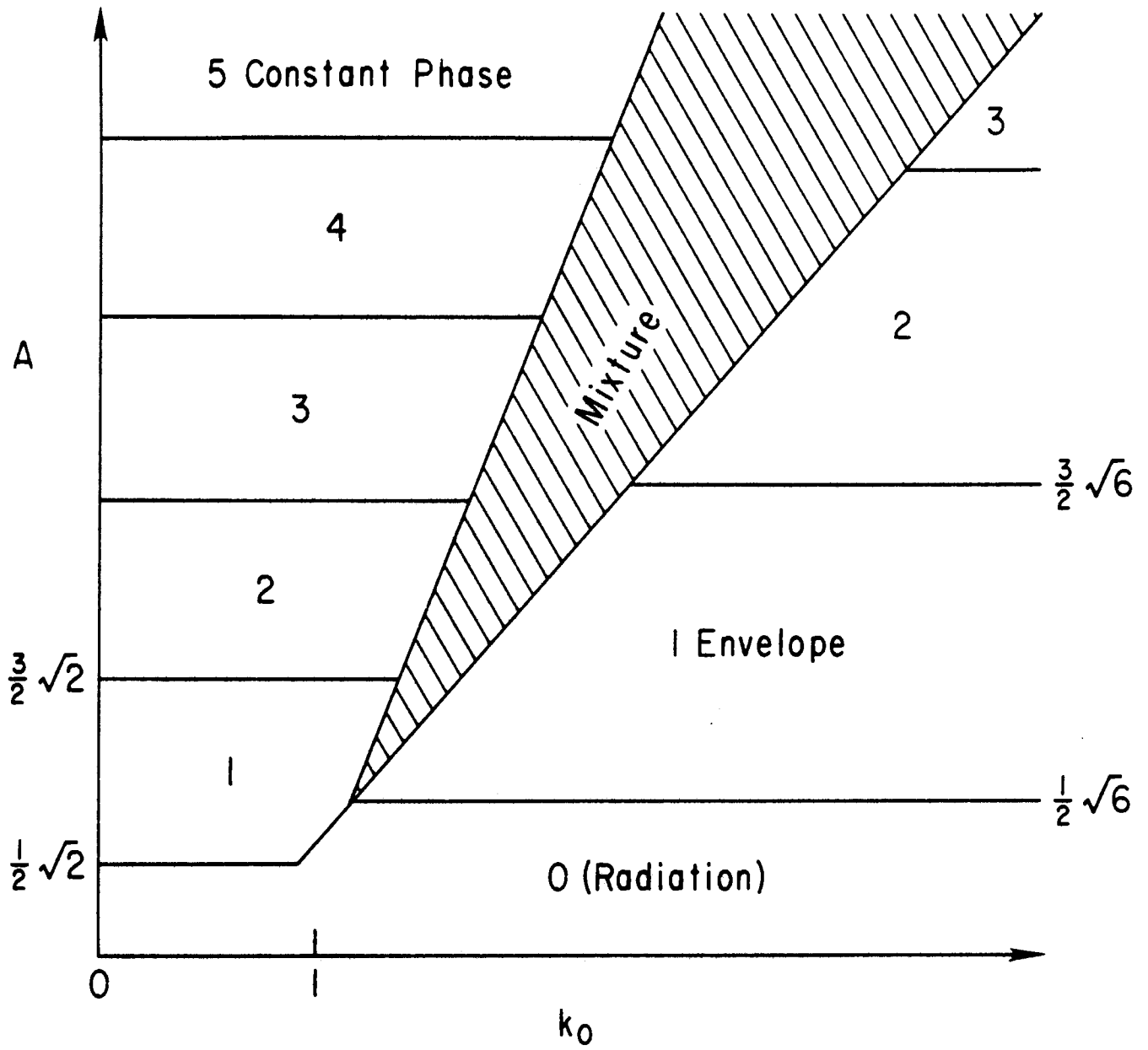


Fig. 5. 782159