

NONLINEAR ORBIT PERTURBATION AND ION HEATING*

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2. NONLINEAR ORBIT PERTURBATION AND ION HEATING

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Introduction

In proposed Tokamak RF heating schemes using waves near the lower hybrid frequency, fields with large $k_{\perp} a_i$ can be generated, either directly by linear conversion into lower hybrid waves or by the parametric decay into lower hybrid waves and ion Bernstein waves. The fields can significantly perturb the motion of an ion, and hence can nonlinearly heat the ions. In this report we present the results of a numerical solution of the orbit equation and give an approximate formula for the ion energy gain.

Basic Equations

We consider an ion in a uniform magnetic field $B_0 \hat{z}$ and perpendicularly traveling electrostatic wave

$$\mathbf{E}(\vec{r}, t) = \hat{y} E_0 \cos(ky - \omega t + \phi). \quad (1)$$

The Lorentz force equation for this particle is

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \frac{q}{m} [\vec{v} \times \vec{B} + \vec{E}(\vec{r}, t)] \\ &= \frac{q}{m} [\vec{v} \times B_0 \hat{z} + \hat{y} E_0 \cos(ky - \omega t + \phi)]. \end{aligned} \quad (2)$$

(XII. PLASMA DYNAMICS)

If we normalize time to $1/\Omega$ ($\Omega = qB_0/m$), length to $1/k$, and velocity to Ω/k , then (2) becomes

$$\frac{d\bar{v}}{dt} = \bar{v} \times \hat{z} + \alpha \hat{y} \cos(y - vt + \phi), \quad (3)$$

where $\alpha = \frac{qE}{m} \frac{k}{\Omega^2}$, and $\nu = \omega/\Omega$. The x-component of (3) is

$$\ddot{x} = \dot{y} \quad (4)$$

and the y-component of (3) is

$$\ddot{y} + \dot{x} = \alpha \cos(y - vt + \phi). \quad (5)$$

We choose initial conditions such that the guiding center is initially at $x = y = 0$, i. e., at $t = 0$:

$$\begin{aligned} \dot{x} &= r \cos \theta & \dot{y} &= r \sin \theta \\ x &= -r \sin \theta & y &= r \cos \theta \end{aligned}$$

Then (4) can be integrated to give

$$\dot{x} = y \quad (6)$$

and (5) may then be written

$$\ddot{y} + y = \alpha \cos(y - vt + \phi). \quad (7)$$

Solution of Equations

We solve (6) and (7) with a predictor-corrector method, which we start with the Taylor's series solution. In Fig. XII-22 we present the result for $\nu = 30$, $\alpha = 30$ with two different initial velocities. Note that in case (a) with the initial velocity = $22 \Omega/k$, the orbits are closed. The energy gain by the ion in this case is very nearly zero. If we increase the initial velocity to $24 \Omega/k$, we see a very different behavior. When the particle is traveling in the y direction, it is trapped momentarily by the wave with which it exchanges a significant amount of energy, which causes the ion orbit to open. Remembering that without a magnetic field the trapping condition for a particle is

$$\left| \frac{\omega}{k} - v_y \right| < v_{tr}, \quad (8)$$

where v_{tr} is the trapping width $(qE/mk)^{1/2}$. We guess from Fig. XII-22 that the condition for trapping in the presence of a magnetic field is that at some point during the cyclotron orbit (8) is satisfied. We can then write the trapping condition as

$$v_o > \frac{E}{k} - v_{tr} \equiv v_{\text{thresh}} \quad (9)$$

where v_o is the initial perpendicular velocity of the ion. With the parameters of Fig. XII-22, Eq. 9 predicts a threshold of $24.4 \Omega/k$ for v_{thresh} , which is close to the threshold observed in Fig. XII-22.

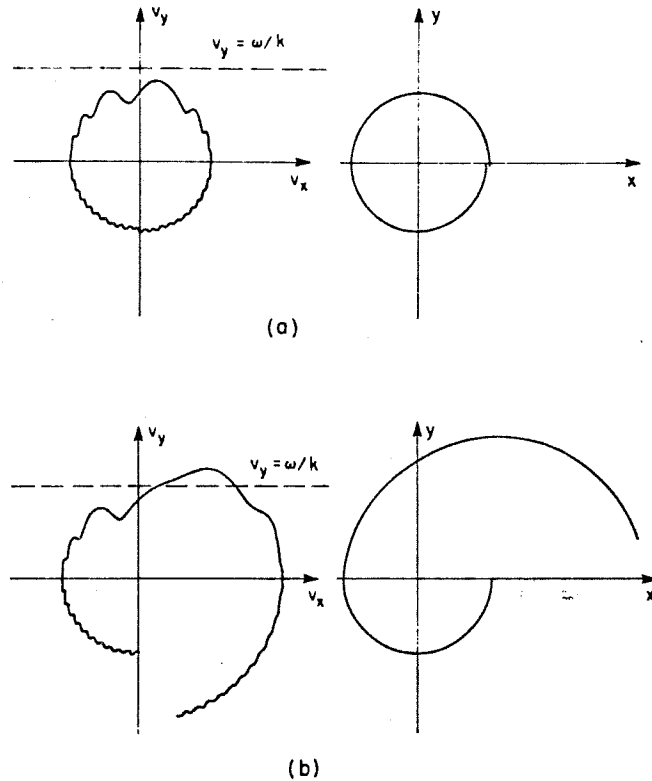


Fig. XII-22. Ion orbits with $\omega = 30 \Omega$ and $a = 30$.
 (a) $v_o = 22 \Omega/k$.
 (b) $v_o = 24 \Omega/k$.

a. Energy Gained by the Ions

Equation 9 is the condition under which the particle exchanges some energy with the wave. We have yet to determine how much it gains. By solving the differential equations with constant ω and E but different values of v_o and ϕ (the electric field phase) we can generate a plot of $\langle \Delta \mathcal{E} \rangle$, the phase average energy gain per particle per cyclotron period, against v_o . Such a plot is shown in Fig. XII-23. We notice a sharp threshold to the energy gain close to v_{thresh} . The curve peaks close to the threshold and drops off at higher velocities. Since the maximum occurs close to the threshold it suggests that the value of the maximum is given by

(XII. PLASMA DYNAMICS)

$$\begin{aligned} \langle \Delta \mathcal{E} \rangle_{\max} &= \frac{1}{2} m \left[\left(\frac{\omega}{k} + v_{\text{tr}} \right)^2 - \left(\frac{\omega}{k} - v_{\text{tr}} \right)^2 \right] \\ &= 2 m \frac{\omega}{k} v_{\text{tr}}. \end{aligned} \quad (10)$$

This just says that a particle that is just trapped will bounce once in the potential well of the field and come out with velocity $(\omega/k + v_{\text{tr}})$. Using (10), we find $\langle \Delta \mathcal{E} \rangle_{\max} = 330 \text{ m}\Omega^2/k^2$

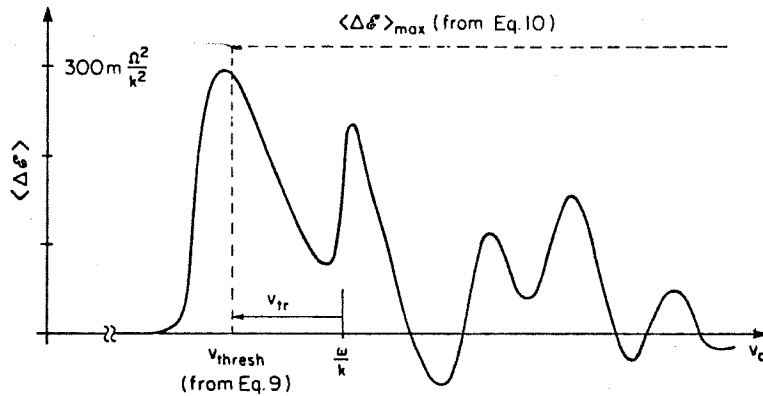


Fig. XII-23. Plot of average energy gain per ion per cyclotron period vs ion velocity for $\omega = 30 \Omega$ and $a = 30$.

for the parameters of Fig. XII-23. This agrees well with the observed value of approximately $300 \text{ m}\Omega^2/k^2$.

We wish to find the energy gain by a perpendicular distribution function $f(v)$ of ions. We integrate $\langle \Delta \mathcal{E} \rangle$ over the distribution functions thus

$$\langle \Delta \mathcal{E} \rangle_{\text{tot}} = n_0 \int_0^{\infty} 2\pi v f(v) \langle \Delta \mathcal{E} \rangle dv. \quad (11)$$

We substitute a Maxwellian for f , so that

$$f = \frac{1}{2\pi v_T} \exp\left(-v^2/2v_T^2\right). \quad (12)$$

Since in (11) we integrate over an exponential function, the main contribution to the integral comes from near $v = v_{\text{thresh}}$. Thus we approximate the function $\langle \Delta \mathcal{E} \rangle$ by a function that is zero for $v < v_{\text{thresh}}$ and equals $\langle \Delta \mathcal{E} \rangle_{\max}$ for $v > v_{\text{thresh}}$ (shown as a dashed line in Fig. XII-23). Performing the integral in (11), we obtain

$$\langle \Delta \mathcal{E} \rangle_{\text{tot}} = n_0 \langle \Delta \mathcal{E} \rangle_{\max} \exp\left(-v_{\text{thresh}}^2/2v_T^2\right). \quad (13)$$

Naturally, the strongest dependence in (13) appears in the exponential, so we predict a large $\langle \Delta \mathcal{E} \rangle_{\text{tot}}$ if $v_{\text{thresh}} \sim v_T$, or from (9)

$$k_{\perp} a_i = \frac{\omega}{\Omega} - \left(\frac{eE}{m} \frac{k_{\perp}}{\Omega^2} \right)^{1/2}. \quad (14)$$

As an example of the strength of this interaction consider a large-amplitude Bernstein decay wave with $E_0 = 10^4$ V/cm, $\omega = 10 \Omega$, $k_{\perp} a_i = 10$ and $n_0 = 10^{14}/\text{cm}^3$, $T_i = 1$ keV and $B_0 = 50$ kG. Evaluating (13), we find that $\langle \Delta \mathcal{E} \rangle_{\text{tot}}/n_0 T = 0.25$, whereas $\frac{1}{4} \epsilon_0 E_0^2/n_0 T_i = 10^{-4}$. Clearly the wave is very strongly nonlinearly damped.