

Link: <http://charles.karney.info/biblio/karney77b.html>

REDUCTION OF THE EQUATION FOR LOWER HYBRID WAVES  
IN A PLASMA TO A NONLINEAR SCHRÖDINGER EQUATION

Charles F. F. Karney

Plasma Research Report 77/10

May 1977

Reduction of the Equation for Lower Hybrid Waves in a Plasma  
to a Nonlinear Schrödinger Equation\*

by

Charles F. F. Karney

Research Laboratory of Electronics and Plasma Fusion Center,  
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

To be presented at the MACSYMA Users' Conference  
University of California, Berkeley, CA  
July 27-29, 1977

\*Work supported by U.S. Energy Research and Development Administration  
(Contract E(11-1)-3070)

# Reduction of the Equation for Lower Hybrid Waves in a Plasma to a Nonlinear Schrödinger Equation\*

by

Charles F. F. Karney

Research Laboratory of Electronics and Plasma Fusion Center,  
Massachusetts Institute of Technology

The equations describing the nonlinear propagation of waves in an anisotropic plasma are rarely exactly soluble. However it is often possible to make approximations that reduce the exact equations into a simpler equation. In this paper we will describe how MACSYMA may be used to make such approximations, and so reduce the equation describing lower hybrid waves into the nonlinear Schrödinger equation which is soluble by the inverse scattering method (ref. 1). It should be pointed out here that we have not used MACSYMA to do the whole problem; rather MACSYMA is used at several stages in the calculation that is otherwise done by hand. This is not to say that MACSYMA could not do the whole problem, just that there is a natural division between calculations that are easiest done by hand, and those that are easiest done by machine.

The equation describing the steady-state two-dimensional electrostatic propagation of lower hybrid waves in a homogeneous magnetized plasma is (refs. 2, 3)

$$K_{\perp} \frac{\partial^2}{\partial x^2} \phi - |K_{\parallel}| \frac{\partial^2}{\partial z^2} \phi + a \frac{\partial^4}{\partial x^4} \phi + b \frac{\partial^4}{\partial x^2 \partial z^2} \phi + c \frac{\partial^4}{\partial z^4} \phi + \frac{\epsilon_0}{4} \alpha_0 \frac{\partial}{\partial x} \left[ \frac{|\nabla\phi|^2}{n_0 T} \frac{\partial}{\partial x} \phi \right] + \frac{\epsilon_0}{4} \beta_0 \frac{\partial}{\partial z} \left[ \frac{|\nabla\phi|^2}{n_0 T} \frac{\partial}{\partial z} \phi \right] = 0, \quad (1)$$

where  $\phi$  is the complex potential and  $x$  and  $z$  are the directions parallel and perpendicular to the magnetic field and the other quantities are constants. (The real potential is  $\text{Re}[\phi \exp(-i\omega t)]$ , where  $\omega$  is the frequency of the wave.) The significance of the terms in equation (1) is as follows: The first two terms (with coefficients,  $K_{\perp}$  and  $|K_{\parallel}|$ ) describe the linear, cold, electrostatic response; they constitute a wave equation and have solutions which propagate along well defined rays (ref. 4). The terms with coefficients  $a$ ,  $b$ , and  $c$  in equation (1) are the corrections due to the finite temperature of the plasma; the effect of these terms is to cause the ray to disperse. The terms on the second line (with coefficients  $\alpha_0$  and  $\beta_0$ ) are due to the nonlinearity of the plasma; these terms arise because in regions where the electric potential is high, the so-called ponderomotive force expels some of the plasma, causing a change in the dielectric properties of the medium.

We wish to reduce equation (1) to a more manageable form. To do this we must decide what type of solution we are looking for. Since we are interested in situations where the nonlinear terms are perturbations to the linear terms, and since wave-like solutions are known for linear problems, interesting solutions to consider are ones of the form

\*Work supported by U.S. Energy Research and Development Administration  
(Contract E(11-1)-3070)

$$\phi(x, z) = \Phi(x, z) \exp(ik_z z - ik_x x), \quad (2)$$

where the wavenumbers  $k_x$  and  $k_z$  are constants and the complex envelope,  $\Phi$ , is slowly varying compared with the exponential. Since we wish to treat the nonlinear terms as a perturbation, we need only consider the leading order contributions to these terms. Thus we can immediately simplify the nonlinear terms since each derivative operator will bring down either  $ik_z$  or  $-ik_x$ ; thus they may be written as

$$\frac{\epsilon_0}{4} \alpha_0 \frac{\partial}{\partial x} \left[ \frac{|\nabla\phi|^2}{n_0 T} \frac{\partial}{\partial x} \phi \right] + \frac{\epsilon_0}{4} \beta_0 \frac{\partial}{\partial z} \left[ \frac{|\nabla\phi|^2}{n_0 T} \frac{\partial}{\partial z} \phi \right] = C |\Phi|^2 \Phi \exp(ik_z z - ik_x x), \quad (3)$$

where  $C$  is a constant. The problem remaining is to reduce the complexity of the linear terms. This we can do by saying that the dispersion has only a weak effect on the solution (in the final equation we will see that the nonlinearity and dispersion are treated as being perturbations of the same order). If we neglect dispersion entirely, then a solution for  $\Phi$  is

$$\Phi(x, z) = \Phi(z - v_g x); \quad (4)$$

i.e. the waves travel along characteristics. We will treat the effects of both dispersion and nonlinearity by letting  $\Phi$  have an explicit  $x$  dependence; thus

$$\phi(x, z) = \Phi(z', x') \exp(ik_z z - ik_x x), \quad (5)$$

where  $z' = z - v_g x$ ,  $x' = x$ . We order the dependencies in equation (5) as follows

$$|ik_x| \gg |v_g \partial/\partial z'| \gg |\partial/\partial x'|, \quad |ik_z| \gg |\partial/\partial z'|. \quad (6)$$

[This ordering is not the only possible one; for instance Morales and Lee (ref. 2) considered the case where  $k_x = k_z = 0$ , and derived a modified Korteweg-deVries equation.]

Rather than using this ordering directly in equation (1), it is more convenient to treat the more general problem. So we re-write the linear terms in equation (1), to give

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\phi + \text{nonlinear terms} = 0, \quad (7)$$

where  $L$  is a polynomial,

$$L(p, q) = K_{\perp} p^2 - |K_{\parallel}| q^2 + ap^4 + bp^2 q^2 + cq^4. \quad (8)$$

Now if  $L(\partial/\partial x, \partial/\partial z)$  operates on equation (5) we may make the replacements

$$\partial/\partial x \rightarrow -ik_x - v_g \partial/\partial z' + \partial/\partial x', \quad \partial/\partial z \rightarrow ik_z + \partial/\partial z'. \quad (9)$$

We may then Taylor expand  $L$  about  $-ik_x$  and  $ik_z$ . This is, of course, most easily done on MACSYMA:

(C1) GRADEF(L(P,Q),L1(P,Q),L2(P,Q))\$

(C2) GRADEF(L1(P,Q),L11(P,Q),L12(P,Q))\$

(C3) GRADEF(L2(P,Q),L12(P,Q),L22(P,Q))\$

Unfortunately MACSYMA has no notation for the derivative of a function with respect to its arguments; thus we use GRADEF to define L1 to denote the derivative of L with respect to its first argument, etc.

(C4) L(P,Q);  
(D4)

L(P, Q)

(C5) %,P=-%I\*KX-ZEPS\*VG\*DZ1+ZEPS^2\*DX1,Q=%I\*KZ+ZEPS\*DZ1;

(D5) L(DX1 ZEPS<sup>2</sup> - DZ1 VG ZEPS - %I KX, DZ1 ZEPS + %I KZ)

Here we have just written L(P,Q), substituted for P (=  $\partial/\partial x$ ) and Q (=  $\partial/\partial z$ ) using equation (9). In order to incorporate the ordering information implied by equation (6) we have introduced the small parameter ZEPS. (ZEPS is chosen rather than, say, EPS, since MACSYMA will treat it as the main variable in CRE forms.) DZ1 and DX1 are used to denote  $\partial/\partial z'$  and  $\partial/\partial x'$  respectively.

(C6) TAYLOR(%,ZEPS,0,2)\$

(C7) LEXPAND:EV(%,L(-%I\*KX,%I\*KZ)=L,  
L1(-%I\*KX,%I\*KZ)=L1,  
L2(-%I\*KX,%I\*KZ)=L2,  
L11(-%I\*KX,%I\*KZ)=L11,  
L12(-%I\*KX,%I\*KZ)=L12,  
L22(-%I\*KX,%I\*KZ)=L22);

(D7)/R/ 1/2 ((DZ1<sup>2</sup> L11 VG<sup>2</sup> - 2 DZ1<sup>2</sup> L12 VG + DZ1<sup>2</sup> L22 + 2 DX1 L1) ZEPS<sup>2</sup>  
+ (- 2 DZ1 L1 VG + 2 DZ1 L2) ZEPS + 2 L)

We carry out the Taylor expansion using TAYLOR, keeping terms up to ZEPS<sup>2</sup>. The result, LEXPAND, is made more compact by making the functional dependence of L on KX and KZ implicit.

Since we are interested in the balance of the nonlinear term, equation (3), against the dispersive part of the linear operator, L, we demand that all but the ZEPS<sup>2</sup> term in D7 vanish identically. (Note that the the ZEPS<sup>2</sup> term contains the dispersive operator,  $\partial^2/\partial z'^2$ .)

(C8) LEXPAND0:COEFF(LEXPAND,ZEPS,0);

(D8)/R/ L

The zeroth order term is just  $L(-ik_x, ik_z)$ . Setting it to zero

$$L(-ik_x, ik_z) = 0 \quad (10)$$

just states that  $k_x$  and  $k_z$  must satisfy the linear dispersion relation.

(C9) LEXPAND1:COEFF(LEXPAND,ZEPS,1);

(D9)/R/ - DZ1 L1 VG + DZ1 L2

(C10) SOLVE(LEXPAND1=0,VG);

SOLUTION

$$(E10) \quad VG = \frac{L2}{L1}$$

$$(D10) \quad [E10]$$

Setting the first order term to zero gives us the expression for  $v_g$ . We recognise E10 as the familiar expression for the group velocity in a dispersive medium,

$$v_g = \frac{L_q}{L_p} \Big|_{p = -ik_x, q = ik_z} \quad (11)$$

(The subscripts  $p$  and  $q$  denote derivatives.)

$$(C11) \text{ LEXPAND2:COEFF(LEXPAND,ZEPS,2);}$$

$$(D11)/R/ \quad 1/2 (DZ1^2 L11 VG^2 - 2 DZ1 L12 VG + DZ1^2 L22 + 2 DX1 L1)$$

$$(C12) \text{ AA:COEFF(LEXPAND2,DX1);}$$

$$(D12)/R/ \quad L1$$

$$(C13) \text{ BB:COEFF(LEXPAND2,DZ1,2);}$$

$$(D13)/R/ \quad 1/2 (L11 VG^2 - 2 L12 VG + L22)$$

Finally we have the order  $ZEPS^2$  terms. Note that it has the form  $A\partial/\partial x' + B\partial^2/\partial z'^2$ , where  $A$  and  $B$  are given by (AA in D12 and BB in D13)

$$A = L_p, \quad B = \frac{1}{2} L_{pp} v_g^2 - L_{pq} v_g + \frac{1}{2} L_{qq} \quad (12)$$

(All the derivatives are evaluated at  $p = -ik_x, q = ik_z$ .) If demand that the  $ZEPS^2$  term balance the nonlinear term, equation (3), we obtain

$$A\Phi_{x'} + B\Phi_{z'z'} + C|\Phi|^2\Phi = 0 \quad (13)$$

If  $A$  is pure imaginary and  $B$  and  $C$  are real (which turns out to be the case) then equation (13) is the nonlinear Schrödinger equation.

The last task is to evaluate the coefficients  $A$  and  $B$ , for  $L$  given by equation (8). Again, in order to get manageable expressions, we will do this approximately. This time we note that the coefficients,  $a$ ,  $b$ , and  $c$  are much smaller than  $K_{\perp}$  and  $K_{\parallel}$ . Again such manipulations are most readily performed on MACSYMA:

$$(C14) \text{ L:KPERP*P^2+KPAR*Q^2+ZDTA*(A*P^4+B*P^2*Q^2+C*Q^4);}$$

$$(D14) \quad (C Q^4 + B P^2 Q^2 + A P^4) ZDTA + KPAR Q^2 + KPERP P^2$$

$$(C15) \text{ (L1:DIFF(L,P),}$$

$$\quad \text{L2:DIFF(L,Q),}$$

$$\quad \text{L11:DIFF(L1,P),}$$

$$\quad \text{L12:DIFF(L1,Q),}$$

$$\quad \text{L22:DIFF(L2,Q),}$$

VG:EV(RHS(E10)))\$

Here we have defined L [see eq. (8)]. The smallness of  $a$ ,  $b$ , and  $c$  is implied by the small parameter ZDTA. We have also defined the various derivatives of L, and VG. The evaluation of A (AA) is straightforward. We Taylor expand AA to obtain the leading term.

(C16) AA:EV(AA,P=-%I\*KX,Q=%I\*KZ,EVAL);

(D16)/R/  $(2 \%I B KX KZ^2 + 4 \%I A KX^3) ZDTA - 2 \%I KPERP KX$

(C17) AA:TAYLOR(AA,ZDTA,0,0);

(D17)/T/  $- 2 KPERP \%I KX + . . .$

I.e.

$$A = -2ik_x K_{\perp}. \quad (14)$$

We repeat this with B (BB).

(C18) BB:EV(BB);

(D18)/R/  $((4 B C^2 Q^8 + (24 A C^2 + 2 B^2 C) P^2 Q^6 + (32 A B C - 2 B^3) P^4 Q^4 + (24 A^2 C + 2 A B^2) P^6 Q^2 + 4 A^2 B P^8) ZDTA$

$+ ((4 C^2 KPERP + 4 B C KPAR) Q^6 + (8 B C KPERP + (24 A C - B^2) KPAR) P^2 Q^4$

$+ ((24 A C - B^2) KPERP + 8 A B KPAR) P^4 Q^2 + (4 A B KPERP + 4 A^2 KPAR) P^6$

$ZDTA^2 + ((4 C KPAR KPERP + B KPAR^2) Q^4 + (6 C KPERP^2 + 6 A KPAR^2) P^2 Q^2$

$+ (B KPERP^2 + 4 A KPAR KPERP) P^4) ZDTA + KPAR^2 KPERP Q^2 + KPAR KPERP^2 P^2$

$/((B^2 P^2 Q^4 + 4 A B P^4 Q^2 + 4 A^2 P^6) ZDTA$

$+ (2 B KPERP P^2 Q^2 + 4 A KPERP P^4) ZDTA + KPERP^2 P^2)$

(C19) BB:TAYLOR(BB,ZDTA,0,1);

(D19)/T/  $\frac{KPAR^2 Q^2 + KPAR KPERP P^2}{KPERP P^2} - ((KPAR^2 B - 4 KPAR C KPERP) Q^4$

$$+ (2 \text{ KPAR KPERP B} - 2 \text{ KPAR}^2 \text{ A} - 6 \text{ C KPERP}^2 \text{ Q} - \text{KPERP}^2 \text{ B P}^2) \text{ ZDTA} \\ /((\text{KPERP}^2 \text{ P}^2) + \dots)$$

Note that we have taken the Taylor series expansion of BB up to order ZDTA. This is because the order ZDTA<sup>0</sup> term is proportional to the order ZDTA<sup>0</sup> terms in L (see D14), and thus when we set L to zero [see eq. (10)] the leading order term will vanish (this is just a reflection of the fact that "cold" contributions to L,  $K_{\perp} \rho^2 - |K_{\parallel}| q^2$ , are non-dispersive). There are a number of ways of incorporating the fact that L = 0 into D19; we chose the following:

(C20) SOLVE(L=0,KPERP);  
SOLUTION

$$(E20) \quad \text{KPERP} = - \frac{(3 \text{ C Q}^4 + 3 \text{ B P}^2 \text{ Q}^2 + 3 \text{ A P}^4) \text{ ZDTA} + \text{KPAR Q}^2}{\text{P}^2}$$

(D20) [E20]

(C21) BB:EV(BB,E20)\$

(C22) BB:TAYLOR(BB,ZDTA,0,1);

$$(D22)/T/ \quad \frac{(3 \text{ C Q}^4 + 3 \text{ B P}^2 \text{ Q}^2 + 3 \text{ A P}^4) \text{ ZDTA}}{\text{Q}^2} + \dots$$

(Note that indeed the coefficient of ZDTA<sup>0</sup> is zero.)

(C23) BB:EV(BB,ZDTA=1,P=-%I\*KX,Q=%I\*KZ,FACTOR);

$$(D23) \quad - \frac{3 (\text{C KZ}^4 + \text{B KX}^2 \text{KZ}^2 + \text{A KX}^4)}{\text{KZ}^2}$$

(Here we have just substituted for p and q.) Thus

$$B = - \frac{3}{k_z^2} (a k_x^4 + b k_x^2 k_z^2 + c k_z^4). \quad (15)$$

Finally a scale transformation on  $\Phi$ ,  $x'$ , and  $z'$  in equation (13) yields

$$i v_{\tau} + v_{\xi\xi} + 2|v|^2 v = 0, \quad (16)$$

a standard form of the nonlinear Schrödinger equation.

We could have saved some steps in the MACSYMA computation had we worked with the explicit form of L (D14) right from the beginning. However this would have had the disadvantage

of confusing the two small parameters in the problem (ZEPS and ZDTA). Also some of the generality of the method would be lost. For instance, a simple extension of the method outlined above to include the effects of a third spatial dimension [which introduces a term,  $K_{\perp} \partial^2 \phi / \partial y^2$  in eq. (1)] is possible (ref. 5). This leads to an unusual generalization of the nonlinear Schrödinger equation,

$$i v_{\tau} + v_{\xi\xi} - v_{\eta\eta} + 2|v|^2 v = 0. \quad (17)$$

The procedure presented here was suggested by the work of Newell and Kaup (ref. 6), who use a more traditional multiple-time-scales approach. The help of F. Y. F. Chu in preparing this paper is gratefully acknowledged.

#### REFERENCES

1. Scott, A. C.; Chu, F. Y. F.; and McLaughlin, D. W.: *The Soliton: A New Concept in Applied Science*, Proc. IEEE, vol. 61, no. 10, Oct. 1973, pp. 1443-1483.
2. Morales, G. J.; and Lee Y. C.: *The Nonlinear Filamentation of Lower-Hybrid Cones*, Phys. Rev. Lett., vol. 35, no. 14, Oct. 1975, pp. 930-933.
3. Karney, C. F. F.; Chu, F. Y. F.; Johnston, G. L.; and Bers, A.: *Solution to Boundary Value Problem for Propagation of Lower Hybrid Waves*, Plasma Res. Rep. 76/11, Research Laboratory of Electronics, Massachusetts Institute of Technology, Jan. 1976
4. Briggs, R. J.; and Parker, R. R.: *Transport of RF Energy to the Lower Hybrid Resonance in an Inhomogeneous Plasma*, Phys. Rev. Lett., vol. 29, no. 13, Sep. 1972, pp. 852-855.
5. Sen, A.; Karney, C. F. F.; Johnston, G. L.; and Bers, A.: *Three-Dimensional Effects in the Nonlinear Evolution of Lower Hybrid Cones*, Paper D17, Proc. Annual Controlled Fusion Theory Conference, San Diego, May 1977.
6. Newell, A. C.; and Kaup, D. J.: *Nonlinear Propagation Along Lower Hybrid Cones*, Paper 4F-15, Bull. American Phys. Soc., vol. 21, no. 9, Oct. 1976, p. 1095.