COMPLEX MODIFIED K-DV EQUATION AND NONLINEAR PROPAGATION OF LOWER HYBRID WAVES*

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Plasma Research Report PRR 78/6
February 1978

To appear in Proceedings of
The Third Topical Conference on Radio Frequency Plasma Heating
California Institute of Technology, CA
January 11-13, 1978

*Work supported by the National Science Foundation (Grant ENG 77-00340) and by the U. S. Department of Energy (Contract EG-77-G-01-4107)

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The nonlinear steady state propagation of lower hybrid waves in a uniform plasma can be described by a "Complex" Modified Korteweg De Vries equation, $v_\tau + (|v|^2 v)_\xi + v_{\xi\xi\xi} = 0$, where v the amplitude of the electric field is complex. This equation is not amenable to analytic solution by the Inverse Scattering Transform method and we solve it numerically. In the limit of a narrow spectrum excitation at the boundary it approximately reduces to a nonlinear Schrödinger equation and we obtain envelope soliton solutions. For broader spectrums we obtain "MKDV type solitons" with constant phases. We discuss these solutions in terms of satisfying the radiation condition inside the plasma and the limitations posed on the choice of "initial" conditions by nonlinear reflections.

In a cold homogeneous plasma the linear steady state propagation of lower hybrid waves is along resonance cones. Treating nonlinear and dispersive (thermal) effects as small perturbations along any single resonance cone the propagation equation can be derived as[1]:

$$v_\tau + (|v|^2 v)_\xi + v_{\xi\xi\xi} = 0 \tag{1}$$

where v is proportional to the electric field amplitude ($\approx \partial\phi/\partial x$), $\xi \sim (x - cz)$ is the stretched characteristic coordinate (c is the ratio of cold group velocities) and $\tau \sim x$ characterises the perturbative effects of nonlinearity and dispersion. If v is assumed real, (1) reduces to the Modified Korteweg-de Vries (MKDV) equation[2] which has soliton solutions. However in general v is complex and then (1) is the correct nonlinear equation to solve. Unlike the MKDV equation, this equation, the "Complex Modified K-dV" (CMKDV) equation, does not appear to have an infinite set of conservation laws and is not amenable to solution by the Inverse Scattering Transform method. In this paper we therefore present the results of numerically integrating (1).

Equation (1) has a conservation law

$$\int_{-\infty}^{\infty} |v(\tau, k)|^2/k \, dk = \text{const.} \tag{2}$$

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where k is the transform variable for ξ. This is just the same as the power flow conservation law $\int_{-\infty}^{\infty} S_x\, dx$. (Note that $S_x(k) \propto |v_k|^2/k$.) Thus when imposing "initial" conditions to (1) at $\tau = 0$ (i.e. $x = 0$), we require that $v(0, k < 0) = 0$ so that all spectral components of the initial pulse carry power into the plasma. This may be achieved by ensuring that $v(0, \xi)$ has the form

$$v(0, \xi) = \frac{1}{2} w(\xi) + \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{w(\xi')}{\xi' - \xi} d\xi' . \tag{3}$$

For the numerical integration of (1) we find it more convenient to choose initial conditions of the form

$$v(0, \xi) = A \operatorname{sech}(\xi) \exp(ik_0\xi). \tag{4}$$

For $k_0 > 1$ this only has a small tail of negative k components which we do not expect to materially effect the results. We note that if $k_0 = 0$, we may obtain the MKDV soliton by letting $A = \sqrt{2}$. The soliton threshold is $A = 1/\sqrt{2}$.

We begin first by looking at $k_0 >> 1$. In this case (1) approximately reduces to the Nonlinear Schrödinger equation (NLSE)[3]. If we make no approximations (1) becomes

$$u_\eta + iu_{\zeta\zeta} + 2i|u^2|u + u_{\zeta\zeta\zeta}/(3k_0) + 2(|u^2|u)_\zeta/k_0 = 0 \tag{5}$$

where $v(\tau, \xi) = \sqrt{6}u(\eta, \zeta) \exp[i(k_0\xi + k_0^3\tau)]$, $\zeta = \xi + 3k_0^2\tau$, $\eta = 3k_0\tau$. Note that in the limit of $k_0 \to \infty$ (when (5) becomes the NLSE) the single soliton solution is obtained when $A = \sqrt{6}$ in (4), i.e. $\sqrt{3}$ times the area of the soliton for the case $k_0 = 0$. We integrate (5) with an initial condition of $u = \operatorname{sech}(\zeta)$. We find that for $k_0 > 2\frac{1}{2}$ the pulse is long-lived and remains sech-like but it has a finite velocity in the η, ζ frame (unlike the case $k_0 \to \infty$). For $k_0 < 2\frac{1}{2}$ the pulse radiply loses its identity, and the NLSE approximation is no longer valid.

This brings us to the other limit to consider, namely $k \sim 1$. (This is important in 2 and 4 waveguide excitations.) We show in Fig. 1 the results of integrating (1) with $A = 4.0$ and $k_0 = 1.2$. Note that the pulse breaks up into "solitons" confirming the results of Kuehl[4]. These solitons have the form

$$v(\tau, \xi) = \exp(i\theta) \sqrt{2} a \operatorname{sech}[a(\xi - \xi_0 - a^2\tau)] , \tag{6}$$

where θ is a constant. This is just the solution of the MKDV equation multiplied by a complex constant. These solutions have the property that quite general initial conditons lead to the formation of solitons at large τ. (Note the characteristic ordering of the solitons in Fig. 1, with the tallest and fastest leading the others.) These solitons also have the property that they are unaffected by collisions with each other (except for a translation of the soliton), when they all have the same phase, θ. However the collision of two solitons with different values of θ, is inelastic, in that some "radiation" is produced. In Fig. 2, we show the collision of two solitons with $a_1 = \sqrt{2}$, $a_2 = 1/\sqrt{2}$, $\theta_1 = 0$, and $\theta_2 = \frac{1}{2}\pi$. We see that after the collision the tallest soliton remains unaffected, while the shorter one has lost some of its amplitude. The phases of these solitons are nearly interchanged, with $\theta_1 = 80°$ and $\theta_2 = 10°$.

The problem with the solution shown in Fig. 1 is that the solitons carry no net power. So that Fig. 1 shows a situation where all the power injected at the boundary ends up being carried by the radiation. The solitons correspond to field structures going off to $\tau = \infty$ in which there are equal amounts of positive and negative power flow. This "steady state" is obviously not accessible in a finite time; so we must ask ourselves what assumptions we have made that causes us this trouble. The answer is that the nonlinear term in (1) can cause internal reflection of the power (by changing the sign of k). In systems where there is internal reflection we must specify a radiation condition at the far end of the system. Taking this point to be $\tau = \tau_1$ we should require that $v(\tau_1, k < 0) = 0$, i.e. power only flows outwards at the far boundary. Note that this condition is violated in Fig. 1. At $\tau = 0$ we should only impose the incident power; thus we should specify $v(0, k > 0)$, but we may not specify $v(0, k < 0)$ which must emerge as part of the solution such that the radiation condition is satisfied at the far end. This is rather difficult to achieve numerically and we do not yet have a "steady state" solution satisfying these boundary conditons.

Summarizing, we understand the nonlinear behaviour for the narrow spectrum excitations ($k \gg \Delta k$) where selfmodulation effects can lead to envelope soliton structures. For $k \sim \Delta k$ the problem is not completely solved but it appears that nonlinear internal reflections can occur. In either case, for 2 or 4 waveguide excitations for tokamak plasmas, practical electric field amplitudes and their spatial extent are likely to be below the threshold condition for soliton formation[5] and the nonlinear reflections would then also be relatively unimportant.

Acknowledgement

We would like to acknowledge Bruce Edwards for much help with the numerical work and Flora Chu for valuable comments.

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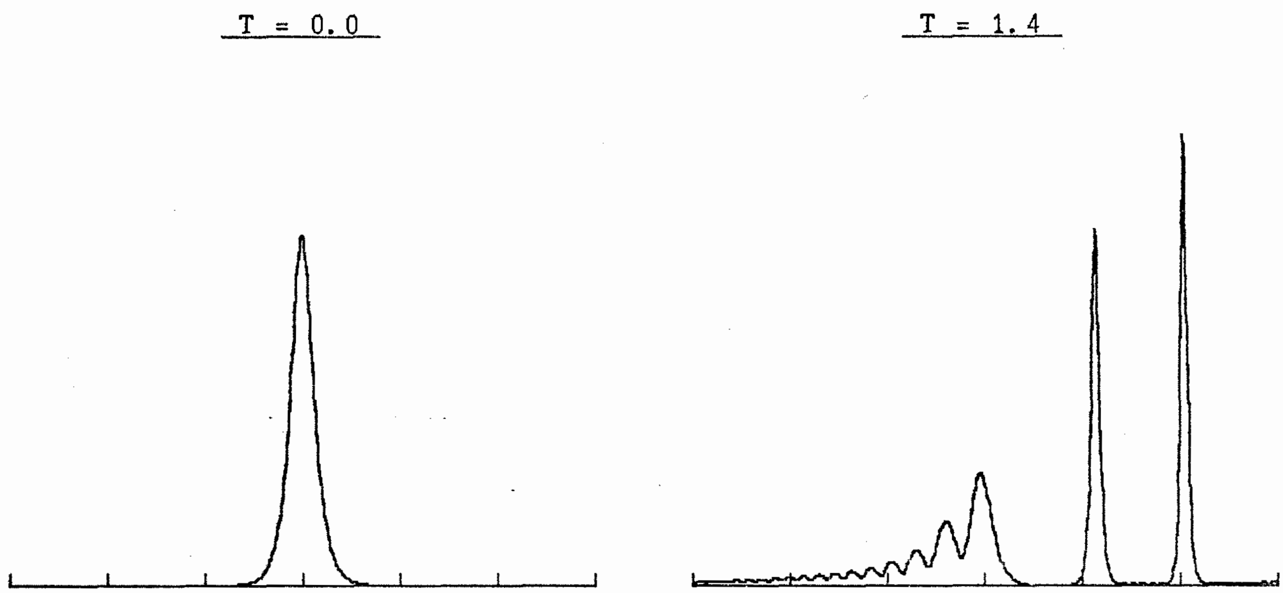

Fig. 1. Evolution of an initial sech pulse with A=4., k_0=1.2 , into "MKDV" type solitons and "radiation".

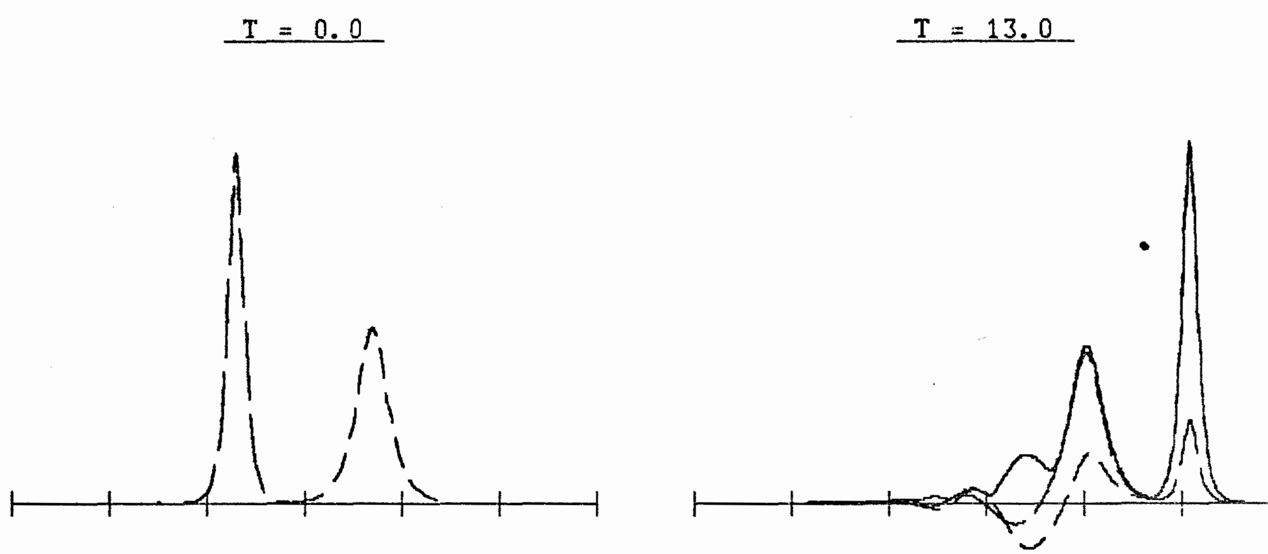

Fig. 2. Collision of two solitons initially placed apart with $a_1 = \sqrt{2}$, $a_2 = 1/\sqrt{2}$, $\theta_1 = 0$, and $\theta_2 = \frac{1}{2}\pi$. At τ=13., the tall soliton has passed through the shorter one but there is some change in the phases and generation of some radiation. Large dashed lines denote Re(v), short dashed lines Im(v) and solid lines abs(v).