STATISTICAL DESCRIPTION OF THE CHIRIKOV–TAYLOR MODEL IN THE PRESENCE OF NOISE

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Abstract. *A review of recent analytical and numerical results concerning the Chirikov–Taylor model is given. It is shown that the presence of noise makes the statistical description of this system unique. The form of the diffusion coefficient is quite different depending on the nature of the dynamical orbits (integrable, stochastic, or accelerator). We have found that in the presence of noise, dynamical averaging, performed numerically, and statistical averaging, performed analytically with the path-integral method, are the same. Some unsolved problems are presented and discussed.*

1. INTRODUCTION

Statistical description of systems far from thermal equilibrium constitutes one of the major unsolved problems of modern physics. The fundamental question in this area is how to derive the statistical properties of such systems based on the equations of motion. This is usually quite a difficult question to answer because it involves a description of dynamical systems for very long times. So far, studies of this kind have been done only for simple systems with a very few degrees of freedom. The purpose of the present chapter is to give a review of recent results related to the Chirikov–Taylor (C–T) model, also called standard mapping,[1,2] in the presence of Gaussian noise.

The C–T model is described by the equations

$$x_{t+1} = x_t + y_{t+1}, \tag{1}$$

$$y_{t+1} = y_t - \varepsilon \sin x_t. \tag{2}$$

Here $t = 0, 1, 2, \ldots, T$ corresponds to a discrete time and $0 \leqslant x \leqslant 2\pi$ is a phase variable. The variable y corresponds to energy,[1] magnetic moment,[3] velocity,[4,5], or other physical property, depending on the physical context. Consider, for example, the motion of a charged particle in a uniform magnetic field B in a cyclotron. Let us apply a periodic potential $U \sin \omega t$ within a small cross section (see Fig. 1). The equations that describe the changes of particle energy W and phase of the wave ϕ every time it passes through the acceleration region can be written in the form[1]

$$W_{i+1} = W_i + eU \sin \phi_i, \tag{3}$$

$$\phi_{i+1} = \phi_i + 2\pi \frac{\omega W_{i+1}}{eBc}. \tag{4}$$

Here e is the charge, and $\omega_c = eBc/W$ is the cyclotron frequency of the relativistic particle. Equations (3) and (4) can be rewritten in dimensionless form (1) and (2) with a parameter $\varepsilon = 2\pi\omega U/cB$. This method is used to heat particles: above stochastic threshold $\varepsilon \gtrsim 1$ the average energy of particles grows as the $\sqrt{t}$.

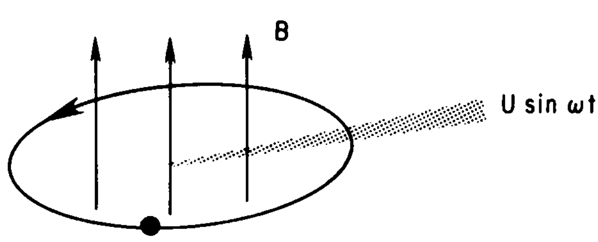

Figure 1. An electron trajectory in a cyclotron. A periodic potential $U \sin \omega t$ is applied in the shaded sector.

Equations (1) and (2) are area preserving. They can be rewritten in Hamiltonian form

$$\frac{dx}{dt} = y, \tag{5}$$

$$\frac{dy}{dt} = -\varepsilon \sum_{n=-\infty}^{\infty} \sin(x - 2\pi nt). \tag{6}$$

Equations (5) and (6) can be used as a simplified model to study, for example, the motion of charged particles in the field of electrostatic plane waves[4, 5] or the behavior of drift orbits in tokamaks in the presence of drift waves.[6]

2. DYNAMICS, DETERMINISTIC ORBITS

First we will review the behavior of a single orbit described by its coordinates x_t, y_t. A few orbits have been mapped in the phase plane xy on Fig. 2, taken from the paper of J. M. Greene.[7] There are two types of orbits: integrable

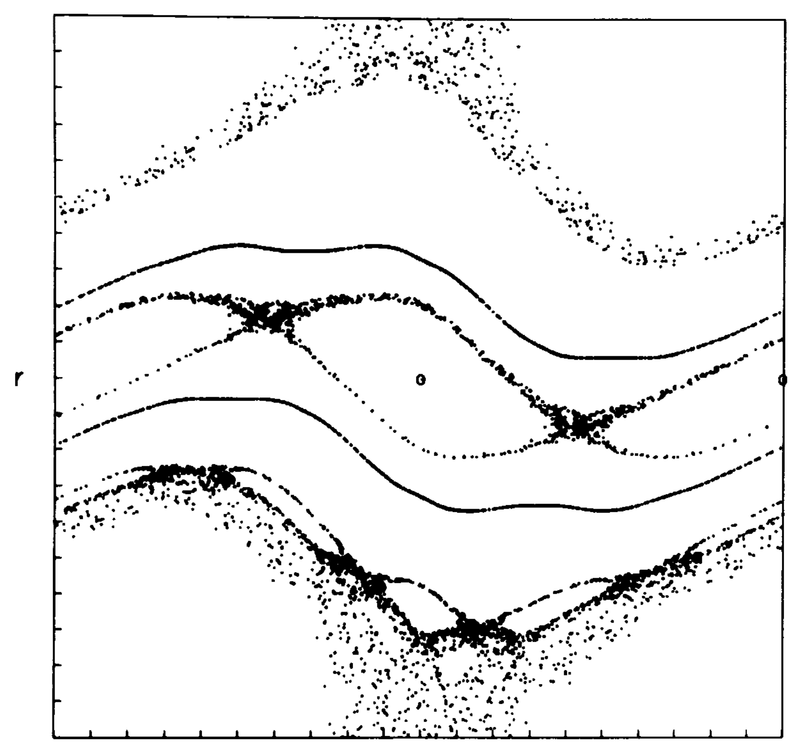

Figure 2. Mapping of orbits for the C–T model, $\varepsilon = 0.97 < \varepsilon_c$ (Ref. 7).

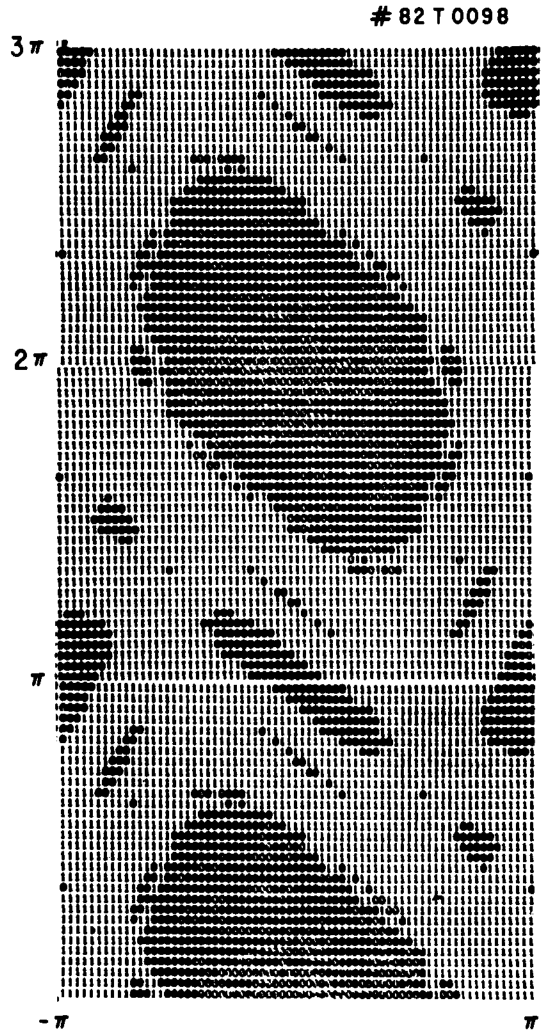

Figure 3. The area covered by a single orbit in the C–T map, $\varepsilon = 1.28$.

orbits, which stay on smooth curves called invariant curves or KAM surfaces, and stochastic orbits, which are area filling. The quantitative measure of stochasticity of an orbit is the maximum Liapunov number λ.[1,2] It defines the rate of exponential divergence of nearby orbits. For stochastic orbits $\lambda > 0$, and for integrable orbits $\lambda = 0$. In the case $\varepsilon \ll 1$ most orbits are integrable, while for $\varepsilon \gg 1$ most orbits are stochastic. For values of $\varepsilon \leqslant \varepsilon_c = 0.9716\ldots$ there exist global invariant curves that divide the phase plane into upper and lower regions as is shown in Fig. 2.[7] These curves are periodic in the y direction because Eqs. (1), and (2) do not change with the transformation $y \to y \pm 2\pi$, $x \to x$. Obviously the existence of such curves bounds the excursion of the orbits in the y direction. At the value of $\varepsilon = \varepsilon_c$ the last of these curves is destroyed, and for some orbits the motion in the y direction becomes unbounded. In Fig. 3 we have plotted the area covered by one such orbit for $\varepsilon = 1.28$.[4] We can define quantitatively the diffusion coefficient for a single orbit by the equation

$$D_{\text{orb}} = \lim_{s \to \infty} \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \frac{1}{2s} (y_{t+s} - y_t)^2. \tag{7}$$

This type of averaging is called dynamic averaging. It requires following the orbit for very long times. We have calculated D_{orb} numerically using Eq. (7) and found that it changes significantly depending on the orbit chosen.[8] For example, if we take an orbit in the dark area of Fig. 3 we obviously get $D_{\text{orb}} \equiv 0$. But for the orbit located in the light area D_{orb} is finite.

Another interesting property of the C–T model is the existence of acceleration orbits.[2] For these orbits y_t increases approximately linearly with time $y_t \approx at$, where the parameter a corresponds to acceleration when y has the meaning of velocity. Acceleration orbits are integrable. In order to find corresponding KAM curves we will reduce the phase space in the y direction to the region $-\pi \leqslant y \leqslant \pi$. A number of orbits have been mapped on Fig. 4 in the

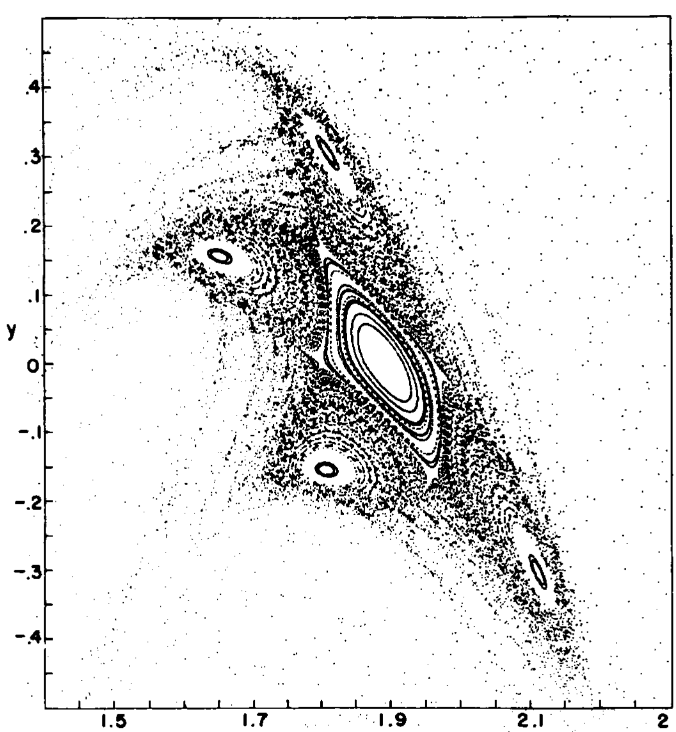

Figure 4. Mapping of accelerator orbits for C–T model, $\varepsilon = 6.615$.

reduced phase space around the stable elliptic point at $x = \cos^{-1}(\varepsilon/4)$, $y = 0$, and $\varepsilon = 6.615$. The large island around this first-order elliptic point is called an accelerator island. There are also four surrounding islands associated with stable elliptic points of fourth order. This whole structure moves in the original phase space with acceleration 2π, and there is a local rotation of the orbits around the elliptic points. Obviously acceleration orbits have $D_{\text{orb}} = \infty$. There are also apparently stochastic orbits filling the regions between the islands. We have calculated numerically[8] one of these orbits and found the diffusion enhanced over the value for stochastic orbits not near accelerator islands by two orders of magnitude. There is also a symmetric accelerator region near $y = 0$ and $x = -\cos^{-1}(\varepsilon/4)$ with acceleration $a = -2\pi$. The stability condition[2] for the elliptic point of the first order is $(2\pi n)^2 \leqslant \varepsilon^2 \leqslant 16 + (2\pi n)^2$. The case shown in Fig. 4 is for $n = 1$. The acceleration is $\pm 2\pi n$. The size of the acceleration island is a very complicated function of ε (Ref. 8), but the upper bound on its size is given by $\Delta x \approx \Delta y \approx 1/\varepsilon$.[2]

3. THE EFFECT OF NOISE

On the other hand there are always some collisions between particles or other noisy processes present in physical systems. Even when they are very small, over a very long time they make a significant contribution to the particle motion. We model them as random variables δx_t or δy_t added to the right-hand side of Eqs. (1) and (2) and assume that they have a Gaussian distribution

$$P(\delta x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(\delta x)^2}{2\sigma}\right].$$

In the presence of noise calculations based on Eq. (7) we show that D is a unique function of ε independent of initial conditions even for a very weak noise. In Fig. 5 we have presented the results of such computations in the case of δx noise present with variance $\sigma = 10^{-3}$. Obviously noise destroys the memory of initial conditions and allows particles to diffuse from one deterministic orbit to another. In the presence of noise all regions in phase space become accessible to the particle. In the case of a deterministic orbit this is not true even in the case $\varepsilon \gg 1$.

To describe the particle motion in the presence of noise we need to introduce the concept of the probability of an orbit. Consider a particle that starts at $x_0 y_0$ at time $t = 0$. Due to the presence of the noise, there is a certain probability for finding the particle at position $x_t y_t$ at time t. For a given orbit, defined by $\{x_t, y_t\}$, $t = 0, 1, \ldots, T$, there exists a probability $P(\{x_t, y_t\})$ that the particle will traverse this orbit. The method of constructing the probability function P is called the path-integral method. It was introduced by Smolokowsky[9] and Einstein[10] to treat Brownian motion. Later Feynman used it to describe quantum mechanical wave functions in terms of classical orbits.[11]

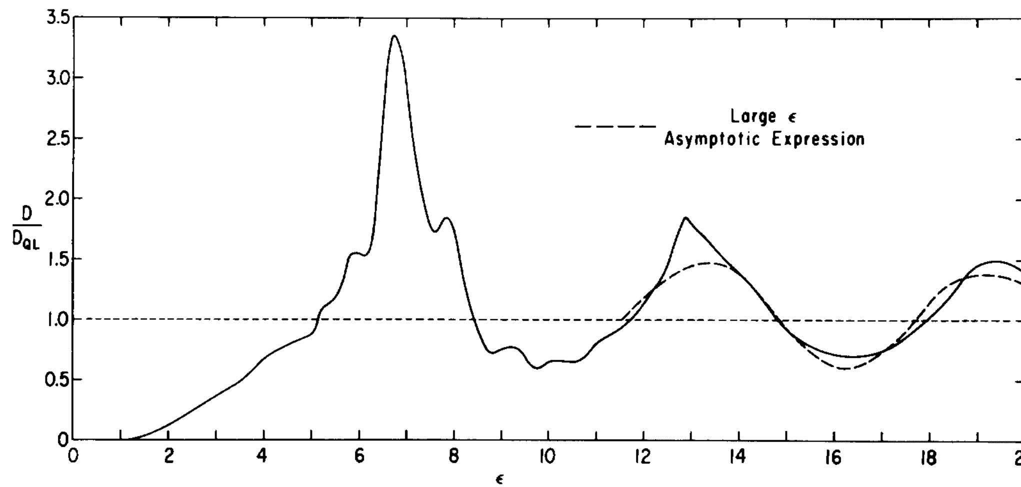

Figure 5. Diffusion in the C–T map. Here $\sigma = 10^{-3}$. The asymptotic curve is from Eq. (10).

For any function of the orbit we define the statistical averaging over the probability P given by

$$\langle f(\{x_t y_t\})\rangle = \prod_{t=1}^{T} \int_0^{2\pi} dx_t \int_{-\infty}^{\infty} dy_t f(\{x_t, y_t\}) P(\{x_t, y_t\}). \tag{8}$$

In this paper we will be particularly concerned with the calculation of the diffusion coefficient, defined by

$$D = \lim_{T\to\infty} \frac{\langle (y_T - y_0)^2 \rangle}{2T}. \tag{9}$$

The diffusion $D(\varepsilon, \sigma)$ is a complicated function of the two parameters ε, σ (see Fig. 5). This complexity is due to the great richness of the dynamical orbits, especially in the region $\varepsilon \approx 1$. However, for large or small ε we were able to explicitly evaluate Eq. (9) and find analytic expressions for the diffusion coefficient D.[5, 12] We will describe now the results of these calculations. Consider first the case $\varepsilon \gg 1$. This is the case when most orbits are stochastic. The first leading term of the asymptotic expansion of D in powers of Bessel functions $J_l(\varepsilon) \sim 1/\sqrt{\varepsilon}$ are given by

$$D = \frac{\varepsilon^2}{2}\left[\tfrac{1}{2} - J_2(\varepsilon)e^{-\sigma} - J_1^2(\varepsilon)e^{-\sigma} + J_2^2(\varepsilon)e^{-2\sigma} + J_3^2(\varepsilon)e^{-3\sigma} + \cdots\right]. \tag{10}$$

Here we have considered only the case of noise in x, δx. In Fig. 6 is shown the comparison of this expression with results obtained by numerically advancing the map. The reason for the good agreement is due to the rapid decay of phase correlations. The term $D_{QL} = \varepsilon^2/4$ corresponds to a random phase approximation often made in quasilinear theory. The Bessel terms result from residual phase correlations of the orbits. Define the phase correlation function

$$\langle \sin x_{t+\tau} \sin x_t \rangle, \tag{11}$$

then the $J_2(\varepsilon)$ term arises from $\tau = 1$, and the J_1^2, J_3^2 terms come from $\tau = 2$ and J_2^2 from $\tau = 3$.[8]

Now consider the case $\varepsilon \ll 1$. In this case most orbits are integrable. In this limit D takes the form

$$D = \frac{\varepsilon^2}{4} \tanh\left(\frac{\sigma}{2}\right). \tag{12}$$

Comparison with numerical results is shown in Fig. 7. In order to calculate Eq. (12), phase correlation functions Eq. (11) must be retained with $1 \ll \tau \lesssim 1/\sigma$. Thus the phase correlations decay only due to the presence of the noise, as

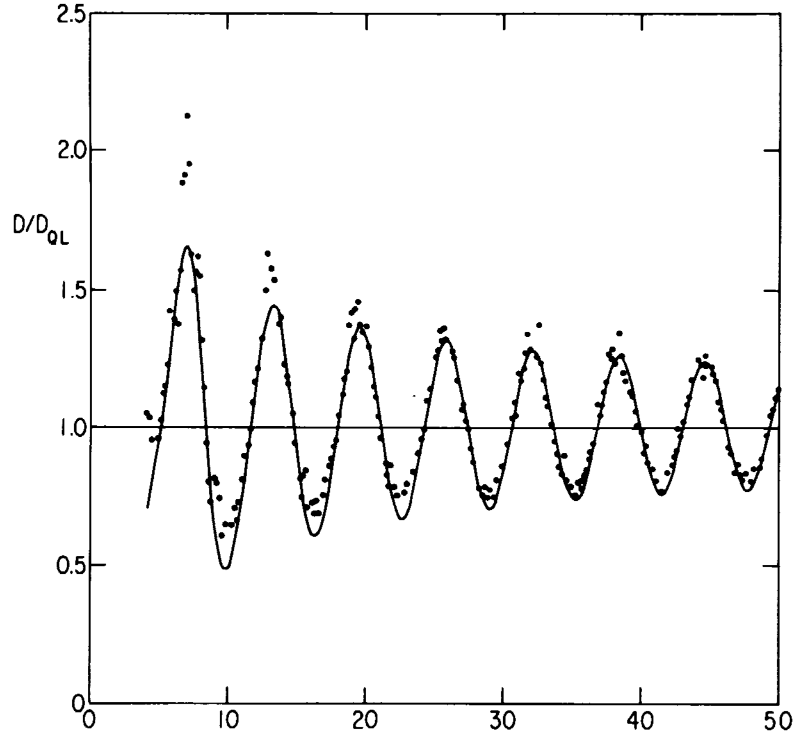

Figure 6. Diffusion in the C–T map. The points are from numerically advancing the map.

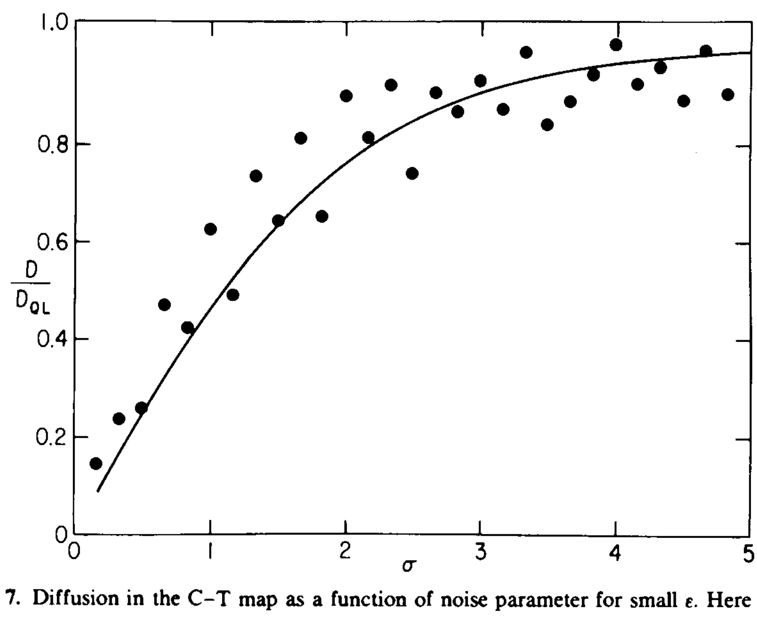

Figure 7. Diffusion in the C–T map as a function of noise parameter for small ϵ. Here $\epsilon = 0.06$.

expected for integrable orbits. Equations (10) and (12) can easily be generalized to include noise δy.[8]

It is interesting to consider the limit of $\sigma \to 0$. In the stochastic regime there is a definite limit

$$D(\varepsilon) = \lim_{\sigma \to 0} D(\varepsilon, \sigma), \tag{13}$$

whereas in the integrable case,

$$\lim_{\sigma \to 0} D(\varepsilon, \sigma) = 0, \tag{14}$$

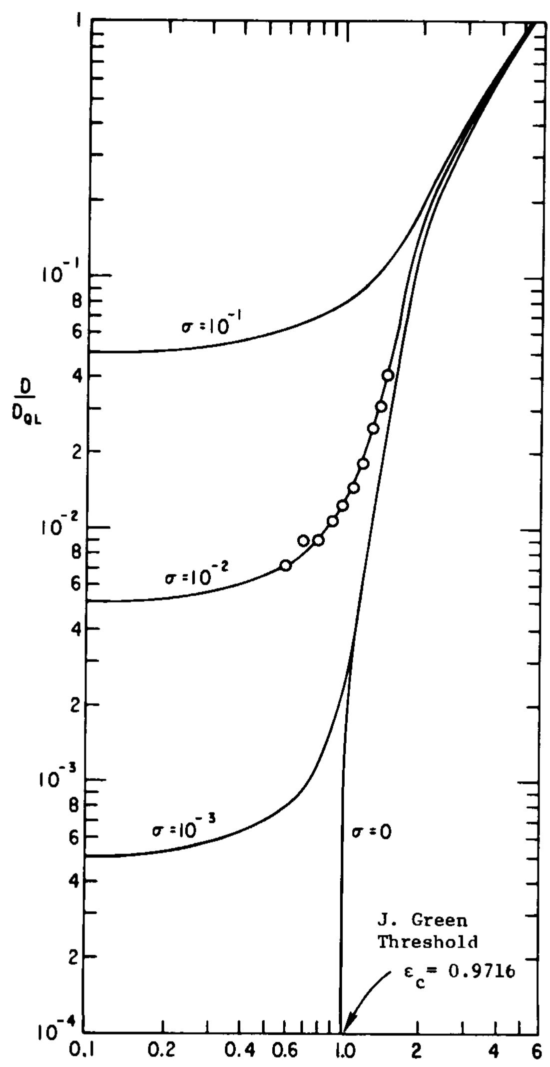

Figure 8. Diffusion in the C–T map for various σ near stochastic threshold.

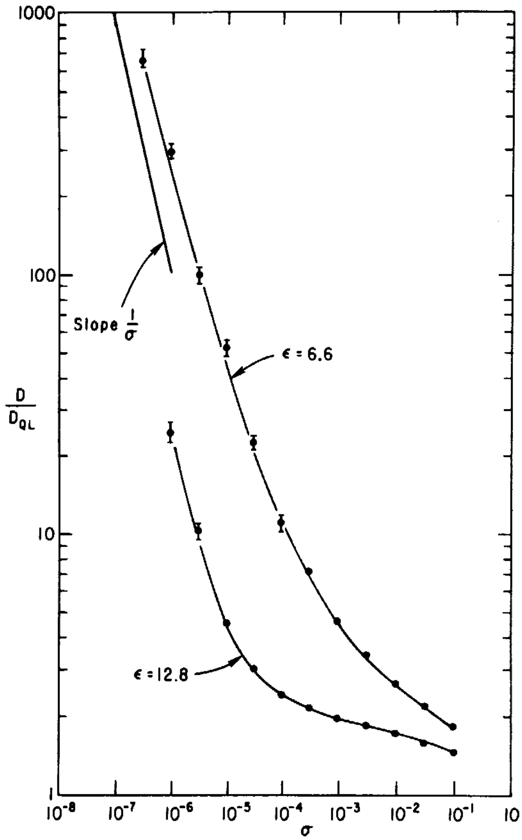

Figure 9. Diffusion in the C–T map in the presence of an accelerator island. For small σ, $D \sim 1/\sigma$.

which is obvious since the diffusion is entirely due to the presence of the noise. The behavior of $D(\varepsilon)$ [Eq. (13)], is quite interesting near diffusivity threshold given by $\varepsilon = \varepsilon_c$, where the last global invariant KAM surface is destroyed. The work[13] shows that the dynamics exhibits universal properties near this threshold, that is, is independent of the periodic function of x in Eq. (2). Our numerical results, shown in Fig. 8, indicate that D has the form

$$D = A(\varepsilon - \varepsilon_c)^{\alpha},$$

and we conjecture that α is a critical exponent.

To conclude this section we consider the effect of noise on accelerator orbits. The effect of the accelerator orbits on diffusion can be estimated as follows. As we saw in Section 2, the acceleration is $a = 2\pi n \approx \varepsilon$. The size of the island is bounded by Δx, $\Delta y \lesssim 1/\varepsilon$. In the presence of noise an orbit will diffuse outside an accelerator island in a time $\tau_{\text{is}} \approx (\Delta x)^2/\sigma$. The time spent outside the island structure is given by $\tau_{\text{out}} \approx 1/\sigma$. The diffusion can be estimated as

$$D_{\text{is}} \lesssim \frac{(\tau_{\text{is}} a)^2}{\tau_{\text{out}}} \lesssim \frac{1}{\varepsilon^2 \sigma}.$$

Thus $D_{\text{is}}/D_{QL} \lesssim 1/\varepsilon^4\sigma$.

The numerical confirmation of this scaling for accelerator islands with $n = 1, 2$ is shown in Fig. 9. We have plotted the ratio of the total diffusion D to D_{QL} for $\varepsilon = 6.6$ and $\varepsilon = 12.8$ where first-order acceleration islands exist. One can see that in the limit of very small σ the scaling $D \approx 1/\sigma$ is verified. We can also conclude that the relative importance of even the largest accelerator island is small unless $\sigma \lesssim 10^{-4}$. Such a small value of σ is necessary in order for an orbit to spend sufficient time in the small island structure. We were not able to calculate D analytically in this case. The main difficulty is that very long time phase correlations must be included.

4. CONCLUSION

The path-integral method proved to be very successful for the statistical description of the Chirikov–Taylor model in the presence of noise.[5, 8, 12] Recently this method has been applied to the variety of problems described by one- and two-dimensional mappings.[14–18] It is also applicable to nonlinear differential equations because they can be written in the form of mappings. To apply the path-integral method to such problems it is necessary to evaluate contributions from many time steps. In general there is no systematic way of doing this, but we have managed to analyze the case of the C–T model in the nearly integrable limit $\varepsilon \ll 1$. An important remaining problem is the development of analytic methods capable of treating similar but more difficult problems such as behavior near the stochastic threshold and in the presence of accelerator orbits.

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