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THREE-DIMENSIONAL EFFECTS IN THE NON-LINEAR PROPAGATION OF LOWER-HYBRID WAVES*

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ABSTRACT. Self-modulation effects can become important for the propagation of lower hybrid waves in plasma, particularly for the high power levels envisioned in r.f. heating schemes. Earlier studies in two dimensions (in the plane defined by the electric field of the pump wave and the background magnetic field) have led to non-linear propagation equations, such as the MKdV or the non-linear Schrödinger equation, which admit multiple-soliton solutions. These could physically manifest themselves by breaking up the resonance cones into filaments with intense localized electric fields and could further lead to localized heating. This problem is studied in three dimensions with the motivation of including two additional physical factors. First, the non-linear effect arising from the $\vec{E} \times \vec{B}$ motion of electrons is included; this leads to an enhancement in the threshold value for the formation of solitons. Secondly, the stability of the two-dimensional solitons to perturbations in the third dimension is studied, and it is found that the third dimension introduces additional dispersive effects which render the solitons unstable to these perturbations.

1. INTRODUCTION

One of the important problems in the lower hybrid heating scheme for tokamak plasmas is the transport of r.f. energy from the waveguides to the interior of the plasma. This has led to extensive theoretical and experimental work on the nature of lower hybrid wave propagation in tokamak-like plasmas. Linear theories have established the basic pattern of r.f. propagation along narrow cones aligned closely to the field lines [1–3]. These so-called "resonance cones" have also been observed experimentally [2, 4, 5].

For high r.f. power, however, non-linear effects can become important and some of these have been studied recently by Morales and Lee [6] and by Kuehl [7]. Basically they take into account the effect of the ponderomotive force arising from the self-action of the wave and self-consistently obtain non-linear wave propagation equations. Depending on the initial assumptions about the nature of the excitation spectrum, these equations are either the modified Korteweg-de-Vries equation (MKdV) [6] or the non-linear Schrödinger equation (NLSE). Since both of these equations may admit multiple-soliton solutions, they conclude that non-linearly the resonance cones will collapse into filaments with intense localized electric fields and cause localized heating either at the surface or in the interior, depending on the plasma conditions. Both these calculations are, however, done in a two-dimensional geometry (i.e. in the plane containing the ambient magnetic field and the electric field of the pump wave) and consider only the component of the ponderomotive force parallel to the magnetic field.

In this paper, we study the lower hybrid propagation problem in three dimensions. The motivation for including this third dimension is two-fold. First of all, a two-dimensional treatment ignores a dominant non-linear effect arising out of the $\vec{E} \times \vec{B}$ motion of the electrons which usually plays an important role in parametric interactions of a lower hybrid wave. In a recent calculation, Kaw et al. [8] have tried to account for this effect by including a model loss term (which mocks up energy transfer into parametrically excited decay waves) into the Morales-Lee treatment and by showing that soliton formation is inhibited at most reasonable values of pump power. In our generalized treatment, we are able to retain the $\vec{E} \times \vec{B}$ contribution to the self-modulation in a consistent manner and find that it raises the threshold for soliton formation by contributing an additional non-linearity. Since, in addition, it introduces coupling between spectral components in the transverse direction, the propagation equations take the form of coupled non-linear

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Schrödinger equations. We will discuss the significance of these equations in a general context. A second important reason for including the third dimension is to investigate the more fundamental question of stability of the solitons obtained from the two-dimensional treatment. In a three-dimensional analysis we find that additional dispersive effects render these solitons unstable to perturbations in the third dimension and thereby prevent the tendency for non-linear filamentation of the lower hybrid cones.

The plan of our paper is as follows: In Section 2, we outline the general derivation of our non-linear evolution equations and point out the effects due to parallel and perpendicular components of the ponderomotive force. These equations are then analysed in two interesting limits in the subsequent sections. In Section 3 we study the effect of the $\vec{E} \times \vec{B}$ contribution on the threshold for soliton formation as well as discuss some of the general implications of the coupled non-linear Schrödinger equations. Section 4 is devoted to the stability analysis of two-dimensional solitons. In Section 5 we summarize our results and discuss their significance for the propagation of lower hybrid waves in tokamak plasmas.

2. DERIVATION OF THE EVOLUTION EQUATIONS

In a warm homogeneous plasma the propagation equation for the monochromatic lower hybrid wave, whose potential is $\phi(\vec{x}) \exp(-i\omega t)$, can be written down as [9]

$$K_\perp \nabla_\perp^2 \phi + K_\parallel \frac{\partial^2 \phi}{\partial z^2} + a_0 \nabla_\perp^4 \phi + b_0 \nabla_\perp^2 \frac{\partial^2 \phi}{\partial z^2} + c_0 \frac{\partial^4 \phi}{\partial z^4}$$
$$- \alpha_0 \nabla_\perp \cdot (n_L \nabla_\perp \phi) - \beta_0 \frac{\partial}{\partial z}\left(n_L \frac{\partial \phi}{\partial z}\right)$$
$$- \gamma_0 \nabla_\perp \cdot (n_L \hat{z} \times \nabla \phi) = 0 \qquad (1)$$

where the magnetic field $\vec{B}_0$ is assumed to lie in the z-direction ($B_0 \hat{z}$). The various constants are given as

$$K_\perp = 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2}, \quad K_\parallel = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}$$

$$a_0 = \frac{1}{4} \frac{\omega_{pe}^2}{\Omega_e^2} \frac{v_{Te}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\omega^2} \frac{v_{Ti}^2}{\omega^2}$$

$$b_0 = -\frac{1}{3} \frac{\omega_{pe}^2}{\omega^2} \frac{v_{Te}^2}{\Omega_e^2} + 2 \frac{\omega_{pi}^2}{\omega^2} \frac{v_{Ti}^2}{\omega^2}$$

$$c_0 = \frac{\omega_{pe}^2}{\omega^2} \frac{v_{Te}^2}{\omega^2} + \frac{\omega_{pi}^2}{\omega^2} \frac{v_{Ti}^2}{\omega^2}$$

$$\alpha_0 = -\frac{\omega_{pe}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\omega^2}, \quad \beta_0 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pi}^2}{\omega^2}, \quad \gamma_0 = \frac{i\omega_{pe}^2}{\omega|\Omega_e|}$$

where ω_{pe} and ω_{pi} refer to the electron and ion plasma frequencies, Ω_e and Ω_i are the electron and ion gyro-frequencies and $v_{Ts} = (3T_s/m_s)^{1/2}$ is the thermal velocity of species s. $K_\perp$ and $K_\parallel$ are components of the cold dielectric tensor and a_0, b_0, c_0 characterize the lowest-order thermal corrections. The last three terms in Eq.(1) represent non-linear effects arising from the ponderomotive action of the wave and n_L is the fractional low-frequency density modulation driven by this force

$$n_L = -\frac{\epsilon_0}{4n_0 T} \left[\alpha_0 |\nabla_\perp \phi|^2 + \beta_0 |\partial\phi/\partial z|^2 + \gamma_0 (\nabla\phi^* \times \nabla\phi)_z\right] \qquad (2)$$

where $T = T_e + T_i$. The terms proportional to α_0 and β_0 are due to the polarization drift and parallel motion of electrons and ions whereas the term proportional to γ_0 arises from the $\vec{E} \times \vec{B}_0$ motion of the electrons. This term would not exist in a two-dimensional analysis (x,z) and arises solely because of the inclusion of the third dimension (y). The magnitude of the ratio of this term to those corresponding to α_0 and β_0 is of the order of $\sim (M/m)^{1/2} k_y/k_x$ for $\omega \sim \sqrt{2}\,\omega_{LH}$. In parametric decay interactions, k_y is the wavevector of the high-frequency sideband and is generally of the same order as k_x of the pump wave, so that the term proportional to γ_0 gives the dominant non-linear contribution. For the problem of self-modulation, however, particularly with a finite waveguide source exciting the wave and far from the source, geometrical considerations make $k_y \ll k_x$, and all the non-linear terms in Eq.(2) are then of comparable magnitude. Further, from the nature of the $\vec{E} \times \vec{B}_0$ term it is clear that a single plane wave-structure in the y-direction does not lead to any effect (the term vanishes identically) and one needs to consider a standing-wave pattern or at least two plane waves in the y-direction. The role of this non-linearity is then to bring about a coupling between the two waves.

We should mention that Eq.(1) is valid away from the edge of the plasma so that electromagnetic corrections can be ignored. It is therefore assumed that the spectrum of excited waves satisfies the accessibility condition ($k_z > k_{z,acc}$) and the waves can penetrate into the plasma. We further restrict ourselves to a narrow spectrum in the z-direction ($\Delta k_z \ll k_z$) so that we can adiabatically expand around the linear plane wave solutions.

We write

$$\phi = \phi_+(x,y,z) \exp(ik_y y + ik_z z - ik_x x)$$
$$+ \phi_-(x,y,z) \exp(-ik_y y + ik_z z - ik_x x)$$

where ϕ_+ and ϕ_- are slowly varying envelopes and the fast varying plane waves satisfy the linear dispersion relation

$$L(-ik_x,\ \pm ik_y,\ ik_z) \equiv -k_\perp^2 K_\perp - k_z^2 K_\parallel + a_0 k_\perp^4$$
$$+ b_0 k_\perp^2 k_z^2 + c_0 k_z^4 = 0, \quad (k_\perp^2 = k_x^2 + k_y^2) \tag{3}$$

We treat the non-linear terms as a perturbation and evaluate them to the lowest order using the linear solutions. From Eq.(2) we obtain

$$n_L = -\frac{\epsilon_0}{4n_0 T}\Big\{k^2(|\phi_+|^2 + |\phi_-|^2) + [\alpha_0(k_x^2 - k_y^2) + \beta_0 k_z^2]$$
$$\times [\phi_+ \phi_-^* \exp(2ik_y y) + \phi_+^* \phi_- \exp(-2ik_y y)]$$
$$- 2\gamma_0 k_y k_x [\phi_+ \phi_-^* \exp(2ik_y y) - \phi_+^* \phi_- \exp(-2ik_y y)]\Big\} \tag{4}$$
$$= n_L^0 + n_L^+ \exp(2ik_y y) + n_L^- \exp(-2ik_y y) \tag{5}$$

[We have used the cold electrostatic dispersion relation, $k^2 = (\alpha_0 k_\perp^2 + \beta_0 k_z^2)$.] Equation (1) can be rewritten as

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi$$
$$- n_L\left(\alpha_0 \nabla_\perp^2 \phi + \beta_0 \frac{\partial^2 \phi}{\partial z^2}\right) - \frac{\partial n_L}{\partial y}\left(\alpha_0 \frac{\partial \phi}{\partial y} + \gamma_0 \frac{\partial \phi}{\partial x}\right) = 0 \tag{6}$$

For the non-linear terms we will retain only those terms proportional to $\exp(\pm ik_y y)$ and neglect the higher harmonic terms proportional to $\exp(\pm 3ik_y y)$. This is a good approximation in our case since we are studying the non-linear evolution of the $\pm k_y$ modes which are driven linearly by the L operator, and the $\pm 3k_y$ modes are linearly non-resonant.

We now use a "multiple-scale" analysis and expand the various derivatives as

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x_0} + \epsilon \frac{\partial}{\partial x_1} + \epsilon^2 \frac{\partial}{\partial x_2} + \cdots \tag{7}$$

and similarly for $\partial/\partial y$ and $\partial/\partial z$. The quantity ϵ is a small parameter characterizing the multiple scales of the variables x, y, z. The physical interpretation of the various orders in Eq.(7) is as follows: The $\partial/\partial x_0$ term represents the fast phase variation ($-ik_x$). The $\partial/\partial x_1$ term gives the slower envelope variation, in the absense of the weak effects of non-linearity and dispersion. Finally, the $\partial/\partial x_2$ term describes the perturbation to the envelope due to non-linearity and dispersion. We carry out a Taylor expansion of the operator L as

$$L \to L\left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial y_0}, \frac{\partial}{\partial z_0}\right)$$
$$+ \epsilon\left(L_1 \frac{\partial}{\partial x_1} + L_2 \frac{\partial}{\partial y_1} + L_3 \frac{\partial}{\partial z_1}\right)$$
$$+ \epsilon^2\left(\frac{L_{11}}{2}\frac{\partial^2}{\partial x_1^2} + \frac{L_{22}}{2}\frac{\partial^2}{\partial y_1^2} + \frac{L_{33}}{2}\frac{\partial^2}{\partial z_1^2}\right.$$
$$+ L_{12}\frac{\partial^2}{\partial x_1 \partial y_1} + L_{13}\frac{\partial^2}{\partial x_1 \partial z_1} + L_{23}\frac{\partial^2}{\partial y_1 \partial z_1}$$
$$\left. + L_1 \frac{\partial}{\partial x_2} + L_2 \frac{\partial}{\partial y_2} + L_3 \frac{\partial}{\partial z_2}\right) + \cdots \tag{8}$$

where the subscripts (1, 2, 3) denote the partial differentiation with respect to the arguments $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, respectively. Equation (6) is now solved recursively, using the expansion (8) and assuming the non-linear contributions to be of order ϵ^2. To order ϵ^0 we recover the linear dispersion relation (3). In the next order, we obtain characteristic equations that enable us to define so-called "wave frames" in which, in the absence of dispersion and non-linearity, the amplitudes ϕ_+ and ϕ_- remain constant. To order ϵ^2 we obtain the following coupled non-linear equations:

$$\tilde{L}_+ \phi_+ + N_+ \phi_+ + M_- \phi_+ = 0 \tag{9a}$$

$$\tilde{L}_- \phi_- + N_- \phi_- + M_+ \phi_- = 0 \tag{9b}$$

where $N_\pm = C|\phi_\pm|^2$ leads to the usual cubic non-linearity associated with the NLSE and $M_\pm = D|\phi_\pm|^2$ is an additional non-linear term that is proportional to the $\vec{E} \times \vec{B}_0$ contribution of the ponderomotive force and that couples the ϕ_+ and ϕ_- modes (C and D are constants) We shall write down the explicit forms of $\tilde{L}_\pm$ as well as define the wave frames for special limiting cases in the next two sections. At this stage we note from the general structure of Eqs (9) that inclusion of the third dimension has introduced two main features: a non-linear coupling between ϕ_+ and ϕ_- and a more complicated dispersive operator $\tilde{L}_\pm$. Our multiple-scale treatment allows us to isolate these two effects and we shall discuss them in the next two sections.

3. TRANSVERSE SPECTRAL COUPLING: ($k_y \neq 0$, $\partial/\partial y_1 \to 0$)

We study first the effect of including the $\vec{E} \times \vec{B}_0$ motion of electrons and the resultant additional non-linear term. To simplify our analysis we shall ignore for the present higher-order effects in the y-direction (i.e. we shall not consider any modulation of the envelope in the y-direction). Accordingly, we set $\partial/\partial y_1 \to 0$ in (8) and solve Eq.(6) recursively in various orders of ϵ. To order ϵ we obtain

$$L_1 \left(\frac{\partial}{\partial x_1} + \frac{L_3}{L_1}\frac{\partial}{\partial z_1}\right) \phi_\pm = 0 \tag{10}$$

which suggests the transformation

$$\zeta = z_1 - g x_1 \tag{11}$$

where

$$g = \left.\frac{L_3}{L_1}\right|_{-ik_x,\, ik_z} = \frac{k_z}{k_x}\,\frac{(K_\parallel - b_0 k_x^2 - 2c_0 k_z^2)}{(-K_\perp + b_0 k_z^2 + 2a_0 k_x^2)} \tag{12}$$

ζ is the characteristic co-ordinate of the wave frame in which ϕ remains constant. To next order (ϵ^2) we get

$$L_3\left(\frac{\partial}{\partial z_2} + \frac{L_1}{L_3}\frac{\partial}{\partial x_2}\right)\phi_\pm + \left(\frac{L_{11}}{2}g^2 + \frac{L_{33}}{2} - gL_{13}\right)\frac{\partial^2 \phi_\pm}{\partial \zeta^2}$$

$$- C|\phi_\pm|^2\phi_\pm - D|\phi_\mp|^2\phi_\pm = 0 \tag{13}$$

where

$$C = \frac{\epsilon_0 k^4}{4n_0 T} \tag{14}$$

$$D = C + \frac{\epsilon_0}{4n_0 T}\{[\alpha_0(k_x^2 - k_y^2) + \beta_0 k_z^2]^2 + 4|\gamma_0|^2 k_y^2 k_x^2\}$$

$$\cong \frac{\epsilon_0 k_y^2 k_x^2}{n_0 T}|\gamma_0|^2 \tag{15}$$

and we have made use of relation (11) to express $\partial/\partial x_1$ and $\partial/\partial z_1$ in terms of $\partial/\partial\zeta$ and we have taken $k_y > (m/M)^{1/2} k_x$. Introducing the normalizations

$$v_\pm = \frac{k_x \phi_\pm}{(4n_0 T/\epsilon_0)^{1/2}} \tag{16}$$

$$\tau = \frac{k}{2gK_\perp}(gx_2 + z_2) \tag{17}$$

$$\xi = \zeta\left(\frac{4n_0 T}{\epsilon_0}\right)^{1/2}\left[\frac{Cd^2}{6k_x^4(a_0 + b_0 d^2 + c_0 d^4)}\right]^{1/2} \tag{18}$$

$$d = +(-K_\perp/K_\parallel)^{1/2} \tag{19}$$

we can rewrite Eq.(13) as

$$i\frac{\partial v_+}{\partial \tau} + \frac{\partial^2 v_+}{\partial \xi^2} + |v_+|^2 v_+ + \tilde{c}|v_-|^2 v_+ = 0 \tag{20a}$$

$$i\frac{\partial v_-}{\partial \tau} + \frac{\partial^2 v_-}{\partial \xi^2} + |v_-|^2 v_- + \tilde{c}|v_+|^2 v_- = 0 \tag{20b}$$

where

$$\tilde{c} = \frac{4k_y^2}{k^2}|\gamma_0|^2 \tag{21}$$

[There is an additional non-thermal "dispersive" contribution to the coefficient in Eq.(18), which arises from the spreading out of the ray over the lower hybrid cone. For small k_y this may be neglected.] Equations (20) are the reduced form of Eqs.(9) in the limit when $\partial/\partial y_1 \to 0$. They describe the non-linear evolution of the two spectral components ϕ_+ and ϕ_- which have oppositely directed k_y components but otherwise satisfy the same linear dispersion relation. The non-linear terms arise from two different components of the ponderomotive force and lead to two distinct effects. The parallel component gives rise to the usual cubic non-linearity and leads to self-focusing or filamentation effects, whereas the $\vec{E} \times \vec{B}_0$ component brings about a mutual coupling between ϕ_+ and ϕ_-.

Equations (20) have been recently analysed in other contexts, e.g. by Berkhoer and Zakharov [10], by Manakov [11] and by Inoue [12] for studying the non-linear behaviour of electromagnetic waves in a plasma. They admit several different kinds of interesting solutions some of which have no analogue in the usual single NLSE. A qualitative picture of these solutions can be obtained by effecting the transformation

$$v_+ = X(\xi) \exp[i\alpha\tau + \theta_1(\xi)] \tag{22a}$$

$$v_- = Y(\xi) \exp[i\beta\tau + \theta_2(\xi)] \tag{22b}$$

(X, Y, θ_1, θ_2 are real) and reducing Eqs (20) to the form

$$\frac{\partial^2 X}{\partial \xi^2} = -\frac{\partial V}{\partial X} \tag{23a}$$

$$\frac{\partial^2 Y}{\partial \xi^2} = -\frac{\partial V}{\partial Y} \tag{23b}$$

where

$$V(X, Y) = \frac{1}{2}\left(\frac{M^2}{X^2} + \frac{N^2}{Y^2}\right) - \frac{1}{2}(\alpha X^2 + \beta Y^2) + \frac{1}{4}(X^4 + 2CX^2Y^2 + Y^4) \tag{24}$$

α, β are arbitrary constants and M and N are constant values of the conserved "angular momenta"

$$M = X^2 \frac{d\theta_1}{d\xi}, \quad N = Y^2 \frac{d\theta_2}{d\xi}$$

From Eqs (23–24) we see that the problem is made equivalent to that for the two-dimensional motion of a particle moving in the potential well $V(X, Y)$ and the various possible orbits in this potential relate to the solutions of Eqs (20). Note that the form of solution in Eqs (22) is not the most general, and it is not yet clear how these solutions relate to the solution of the more general problem where $v_\pm(\tau = 0, \xi)$ are given. Orbits that go off initially (at $\xi = -\infty$) from the point $X = Y = 0$ and return to this point at $\xi = \infty$ correspond to soliton solutions of Eqs (20). This requires that the orbit starts off in a precise direction (where $\partial V/\partial X = \partial V/\partial Y = 0$). In general, however, the orbits do not return to the origin and the particle gets "entangled" in the potential well. The solutions of Eqs (20) corresponding to such orbits have been called "dispersive shock" waves and resemble "envelope shock" waves in their structure, but their character is non-dissipative. Generally, these solutions have the form of an irregular oscillatory non-periodic wave train solution.

For the lower hybrid propagation problem with a finite source excitation, boundary conditions rule out all but the soliton solutions of Eqs (20). This is because a finite source necessitates a finite action (i.e. $\int |v_\pm|^2 d\xi$ = finite) which the dispersive shock solutions fail to satisfy. This is quite analogous to the choice of soliton solutions over conoidal wave solutions for a single NLSE. The soliton solutions of Eqs (20) have been studied by Manakov [11] using the inverse-scattering formalism (for the case $\tilde{c} = 1$). Specifically, he looked at the interaction of N solitons and found that the amplitude and velocity of interacting solitons do not change but the phase has a discontinuity after a collision. This can also be deduced from the fact that both v_+ and v_- separately conserve their "energies" or areas, i.e. from Eqs (20) we obtain

$$\frac{d}{d\tau}\int |v_+|^2 d\xi = \frac{d}{d\tau}\int |v_-|^2 d\xi = 0 \tag{25}$$

For our finite source problem, the situation is further simplified by the fact that $v_+ = v_-$ since the spectrum in the y-direction is symmetric. One can therefore analyse any one of Eqs (20) as a single NLSE and use the standard Zakharov-Shabat formalism [13]. The only additional feature introduced by the $\vec{E} \times \vec{B}_0$ term is to increase the magnitude of the non-linear term, approximately by a factor of 2 (since $\tilde{c}$ is of the order unity). This manifests itself in an increase of the threshold condition for soliton formation by a factor of $\sqrt{2}$.

4. TRANSVERSE ENVELOPE MODULATION: ($k_y \to 0$, $\partial/\partial y_1 \to 0$)

We shall now study the important aspect of stability of the two-dimensional solitons to perturbations in the third dimension [14, 15]. We shall therefore retain modulations of the envelope in the y-direction ($\partial/\partial y_1 \neq 0$). However, we shall ignore fast variations in the y-direction ($k_y \to 0$), since we are basically working with two-dimensional solutions. The limit $k_y \to 0$ also eliminates the $\vec{E} \times \vec{B}_0$ term whose physical contribution we have studied in the earlier section. This greatly simplifies our analysis, since we no longer have to work with coupled equations.

Since to lowest-order L is quadratic in k_y, we can set $L_2 = L_{12} = L_{23} = 0$. The final non-linear equation (to order ϵ^2) then reduces to

$$i\frac{\partial v}{\partial \tau} + \frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \eta^2} + 2|v|^2 v = 0 \tag{26}$$

where

$$v = \frac{k_x \phi}{(4n_0 T/\epsilon_0)^{1/2}} \tag{27}$$

$$\eta = \frac{k}{(2K_\perp)^{1/2}} y_1 \tag{28}$$

and the other normalizations are as defined earlier. Equation (26) is the three-dimensional generalization of the NLSE for the slowly varying envelope of lower hybrid waves. In contrast to the usual NLSE, it does not admit soliton solutions due to the dispersive effect of the additional term. We shall study this effect by using Eq.(26) to carry out a linear stability analysis of two-dimensional soliton solutions. We let

$$v = v_0(\xi) + \delta v(\xi) \cos(k_\eta \eta) \tag{29}$$

where $\delta v \ll v_0$. Substituting in (26), the equation for v_0 is the usual NLSE:

$$i\frac{\partial v_0}{\partial \tau} + \frac{\partial^2 v_0}{\partial \xi^2} + 2|v_0|^2 v_0 = 0 \tag{30}$$

for which a travelling-wave solution can be written down as

$$v_0 = A\,\mathrm{sech}(A\zeta) \exp\left[i\frac{1}{2}u_e(\xi - u_c\tau)\right] \tag{31}$$

where $\zeta = \xi - u_e\tau$ and $A = \frac{1}{2}(u_e^2 - 2u_e u_c)^{1/2}$. Next, the linearized equation for δv is

$$i\delta v_\tau + \delta v_{\xi\xi} + k_\eta^2 \delta v + 4|v_0|^2 \delta v + 2v_0^2 \delta v^* = 0 \tag{32}$$

Writing

$$\delta v = [f(A\zeta) + ig(A\zeta)] \exp\left[i\frac{1}{2}u_e(\xi - u_c\tau) + \gamma\tau\right] \tag{33}$$

where f, g, and γ are real, and separating the real and imaginary parts, Eq.(32) can be reduced to the following coupled set of equations:

$$f_{zz} + 6\,\mathrm{sech}^2 z f - (1 - \kappa^2) f = \Gamma g \tag{34a}$$

$$g_{zz} + 2\,\mathrm{sech}^2 z g - (1 - \kappa^2) g = -\Gamma f \tag{34b}$$

where the following normalizations have been used:

$$z = A\zeta \tag{35}$$

$$\Gamma = \gamma/A^2 \tag{36}$$

$$\kappa = k_\eta/A \tag{37}$$

We now need to solve the eigenvalue problem characterized by Eq.(34) to study the dependence of the growth rate, Γ, on the perturbation wave numbers κ. For $\kappa \ll 1$, a general perturbative method has been developed by Zakharov and Rubenchik [16] and we shall closely follow their analysis from here on.

Equations (34) can be rewritten as

$$(F + \kappa^2) f = \Gamma g \tag{38a}$$

$$(G + \kappa^2) g = -\Gamma f \tag{38b}$$

where F and G are self-adjoint operators defined as

$$F = \frac{d^2}{dz^2} + 6\,\mathrm{sech}^2 z - 1 \tag{39a}$$

$$G = \frac{d^2}{dz^2} + 2\,\mathrm{sech}^2 z - 1 \tag{39b}$$

We wish to solve Eqs (38) perturbatively, about the solution in the limit $\kappa \to 0$. Assuming that $\Gamma = O(\kappa)$ (which we show below), the leading-order solutions are

$$\begin{bmatrix} f^+ \\ g^+ \end{bmatrix} = \begin{bmatrix} \Gamma f_0^+ \\ g_0^+ \end{bmatrix} \quad (40a)$$

$$\begin{bmatrix} f^- \\ g^- \end{bmatrix} = \begin{bmatrix} f_0^- \\ -\Gamma g_0^- \end{bmatrix} \quad (40b)$$

where

$$f_0^+ = \frac{1}{2}\,\mathrm{sech}\,z(1 - z\tanh z),\quad g_0^+ = \mathrm{sech}\,z \quad (41a)$$

$$f_0^- = -\,\mathrm{sech}\,z\tanh z,\quad g_0^- = \frac{1}{2}\,z\,\mathrm{sech}\,z \quad (41b)$$

The superscripts $\pm$ are to distinguish the functions as even or odd in z, respectively. Note that to leading order we retain terms proportional to $\Gamma[= O(\kappa)]$, since we wish to keep the effect of the coupling even in the limit $\kappa \to 0$. Also the perturbation expansion will be in terms of the much smaller quantity, κ^2. Functions (41) also satisfy the following relations:

$$Ff_0^+ = g_0^+,\quad Gg_0^+ = 0 \quad (42a)$$

$$Ff_0^- = 0,\quad Gg_0^- = f_0^- \quad (42b)$$

For $\kappa^2 \ll 1$ we write the solutions for f and g as a series in κ^2 about Eq.(40):

$$\begin{bmatrix} f^- \\ g^- \end{bmatrix} = \begin{bmatrix} f_0^- + f_1^- + f_2^- + \ldots \\ -\Gamma(g_0^- + g_1^- + g_2^- + \ldots) \end{bmatrix} \quad (43)$$

We also express Γ^2 as a series in κ^2:

$$\Gamma^2(\kappa^2) = \Omega_1^- + \Omega_2^- + \ldots \quad (44)$$

Substituting Eqs (43) and (44) in (38) we obtain to first order in κ^2

$$FGg_1^- = -\Omega_1^- g_0^- - \kappa^2(F+G)g_0^- \quad (45)$$

We multiply the above equation by f_0^- and integrate over all space. Then making use of relations (42) it is easy to obtain by simple transformations

$$\Omega_1^- = -\kappa^2 \frac{\langle f_0^- | f_0^- \rangle}{\langle f_0^- | g_0^- \rangle} = \frac{4}{3}\kappa^2 \quad (46)$$

where $\langle a|b \rangle \equiv \int_{-\infty}^{\infty} ab\,dz$. By a similar analysis on the even mode we can obtain

$$\Omega_1^+ = -\kappa^2 \frac{\langle g_0^+ | g_0^+ \rangle}{\langle g_0^+ | f_0^+ \rangle} = -4\kappa^2 \quad (47)$$

From Eq.(46), we see that the odd mode gives a positive growth implying instability of the two-dimensional soliton to any asymmetric perturbations in the third direction. In fact, as Zakharov and Rubenchik [16] demonstrate, this is a very general property of equations of the type of Eq.(26). The presence of the additional dispersive term $\partial^2 v/\partial\eta^2$ always predicts instability and the sign of this term determines whether the even or the odd modes are unstable. Equation (47) gives an imaginary value for γ which contradicts the assumption made in separating Eq.(32) into real and imaginary parts. So we will concentrate on the odd mode.

We can evaluate the next-order term in Eq.(44) by making use of the following relation which can be derived simply:

$$Gg_1^- = \frac{1}{6}\kappa^2\, z\,\mathrm{sech}\,z\,(z\tanh z - 5) \quad (48)$$

We then obtain:

$$\Omega_2^- = -\kappa^4 + 2[\kappa^2\langle f_0^- | Gg_1^- \rangle + \Omega_1^- \langle g_0^- | Gg_1^- \rangle]$$

$$= -\frac{4}{9}\kappa^4\left(\frac{\pi^2}{3} - 1\right) \cong -1.02\,\kappa^4 \quad (49)$$

Adding Eqs (46) and (49) we can write:

$$(\Gamma^-)^2 = \frac{4}{3}\kappa^2 - 1.02\,\kappa^4 + O(\kappa^6) \quad (50)$$

Extending Eq.(50) to κ of order unity we see that the growth rate decreases for κ greater than a critical value. The maximum in Γ^2 occurs around $\kappa^2_{max} \cong \frac{2}{3}$ and is of order κ^2. Thus $\gamma = Ak_\eta$, and so in terms of unnormalized quantities, we may obtain an estimate of the e-folding distance of growth in x as

$$L_g \cong (2K_\perp)^{1/2} \frac{\lambda_y}{2\pi \mathscr{E}^{1/2}} \quad (51)$$

where λ_y is the perturbation wavelength in the y-direction and $\mathscr{E} = A^2 = (\epsilon_0 E^2)/(4n_0 T)$ is a measure of the soliton amplitude. Equation (51) is only valid for $\kappa \leqslant 1$ (approximately) or

$$\lambda_y \geqslant \lambda_{y,\min} = 2\pi (2K_\perp/\mathscr{E})^{1/2}/k \quad (52)$$

We thus obtain

$$L_{g,\min} = 2K_\perp \frac{\lambda_{\min}}{2\pi \mathscr{E}^{1/2}} \cong 2K_\perp \frac{1}{\mathscr{E} k} \quad (53)$$

Thus the tendency for two-dimensional solitons to form and propagate stably is opposed by this process of perpendicular break-up and it proceeds faster for higher powers. It is unlikely therefore that the lower hybrid resonance cones will degenerate into two-dimensional soliton structures with intense electric field concentrations.

Our three-dimensional Eq.(26) also occurs in other physical situations, e.g. in the description of deep-water waves [17] (with no surface tension). To the best of our knowledge, no soliton solutions have been found for this equation. We have obtained, however, a self-similar solution to Eq.(26), namely

$$v = \frac{B}{\tau} \exp \left[\frac{i(\xi^2 - \eta^2)}{\tau} - 2i \frac{BB^*}{\tau} \right] \quad (54)$$

which exhibits rapid oscillations and a slow decay as a function of τ. The amplitude B is approximately uniform with respect to the variables ζ, η and τ. Such a solution could possibly describe the non-linear state of the lower hybrid cones in the interior of the plasma.

5. SUMMARY

We have studied the non-linear propagation of lower hybrid waves in three dimensions. The principal motivation for including the third dimension was to examine the effect of two additional physical factors. First of all, we are able to include the non-linear effect arising out of the $\vec{E} \times \vec{B}_0$ motion of electrons. This is found to lead to an enhancement in the threshold value for the formation of solitons by approximately a factor of $\sqrt{2}$, over earlier calculations done in two dimensions. The $\vec{E} \times \vec{B}_0$ non-linearity also brings about a coupling between spectral components with oppositely directed k_y components (but satisfying the same linear dispersion relation) and the resultant non-linear equations are coupled NLSE's. These, in general, have various interesting solutions (e.g. dispersive shock waves) which could physically manifest themselves in situations where boundary conditions are not as restrictive as in the waveguide excitation problem.

Secondly, we investigate the stability of two-dimensional solitons to perturbations in the third dimension. We find that the third dimension introduces additional dispersive effects which render the solitons unstable to these perturbations. The non-linear development of this instability has yet to be studied. However, there does not appear to be any soliton-type solutions to our generalized three-dimensional non-linear equation. We suggest an exact non-linear solution of the self-similar kind.

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