LONG-TIME CORRELATIONS IN STOCHASTIC SYSTEMS*

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Abstract

In recent years, there has been considerable interest in understanding the motion in Hamiltonian systems when phase space is divided into stochastic and integrable regions. This paper studies one aspect of this problem, namely, the motion of trajectories in the stochastic sea when there is a small island present. The results show that the particle can be stuck close to the island for very long times. For the standard mapping, where accelerator modes are possible, it appears that the mean squared displacement of particles in the stochastic sea may increase faster than linearly with time indicating non-diffusive behavior.

Introduction

Many important problems in physics are described by Hamiltonians of two degrees of freedom. Examples are the motion of a charged particle in electrostatic waves, the motion of a charged particle in various magnetic confinement devices, the acceleration of a particle bouncing between a fixed and an oscillating wall, the wandering of magnetic field lines, etc. In such systems, there is usually a range of parameters (normally when the coupling between the two degrees of freedom is large), where the motion in nearly the whole of phase space is stochastic. Such behavior is seen for instance in the standard mapping (Chirikov, 1979),

$$r_t - r_{t-1} = -k \sin\theta_{t-1}, \qquad \theta_t - \theta_{t-1} = r_t.$$

When k is large most of phase space is stochastic. However, there may still be small islands present; stochastic trajectories can wander close to these islands and remain there for a long time leading to unexpectedly long correlations. The effect of these correlations can be dramatic. The simplest approximation for the diffusion coefficient,

$$\mathcal{D} = \lim_{t\to\infty} \frac{\langle (r_t - r_0)^2 \rangle}{2t}$$

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(where the average is over some appropriate ensemble), is given by assuming that the phase θ is a random variable in the equation for r. This gives the "quasi-linear" result $\mathcal{D} = \mathcal{D}_{\rm ql} = \frac{1}{4}k^2$. However, a numerical determination (Karney *et al.*, 1982) of the diffusion at $k = 6.6$, where the random phase approximation might be expected to be accurate, gave $\mathcal{D}/\mathcal{D}_{\rm ql} \sim 80$. At this value of k there is an island ("accelerator mode") present in the stochastic sea. This leads to long-time correlations in the acceleration of the particle and an enhanced diffusion coefficient.

In this paper, we examine more closely the effect these islands have on a stochastic trajectory. As far as determining the effect on the correlation function, this involves determining how "sticky" the island is. Given that the stochastic trajectory comes within a certain distance of the boundary of the island, how long do we expect it to stay close to the island? This approach is inspired by work of Channon and Lebowitz (1981) on the correlations of a trajectory in the stochastic band trapped between two KAM surfaces in the Hénon map. Similar work has been carried out on the whisker map by Chirikov and Shepelyansky (1981); this work is being extended by B. V. Chirikov and F. Vivaldi. The work reported herein is described in more detail by Karney (1983).

Derivation of Mapping

Far into the stochastic regime for a general mapping, the islands which appear via tangent bifurcations are very small and exist only for a small interval in parameter space. This allows us to approximate them by a Taylor expansion in both phase and parameter space about the tangent bifurcation point retaining only the leading terms. This was carried out by Karney *et al.* (1982) where the resulting mapping was reduced to a canonical form

$$Q: \qquad y_t - y_{t-1} = 2(x_{t-1}^2 - K), \qquad x_t - x_{t-1} = y_t.$$

Here K is proportional to $k - k_{\rm tang}$ ($k_{\rm tang}$ is the parameter value where the tangent bifurcation takes place) and x and y are related to the original phase space coordinates by a smooth transformation. The quadratic mapping Q represents an approximation of the general mapping close to the point of tangent bifurcation. For $0 < K < 1$, this mapping has stable (elliptic) and unstable (hyperbolic) fixed points at $(x, y) = (\mp\sqrt{K}, 0)$, respectively. The elliptic fixed point is usually surrounded by integrable trajectories (KAM curves) which define a stable region (the island) in which the motion is bounded. An example of island structure is shown in Fig. 1 for $K = 0.1$ (the value of K at which extensive numerical calculations have been carried out).

Referring to the islands shown in Fig. 1, consider a particle which at $t = 0$ is close to, but outside, the islands. Initially, the particle will stay close to the islands; however as we let $t \to \pm\infty$, we find $(x, y) \to (\infty, \pm\infty)$. It is just such trajectories we are interested in, because they correspond to particles in

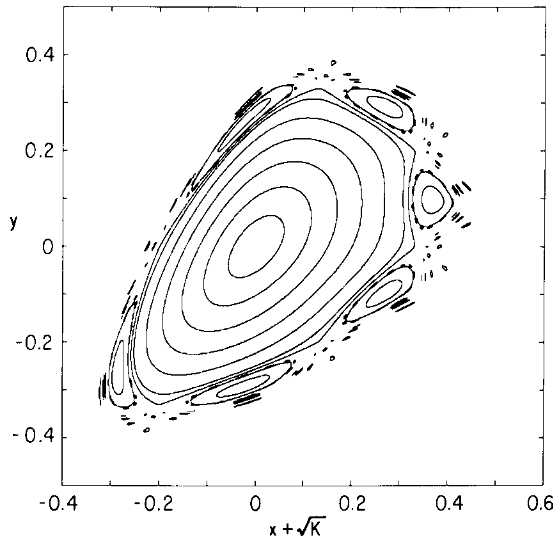

Fig. 1. Some islands of the quadratic map Q for $K = 0.1$.

the stochastic region of the general mapping approaching the islands, staying there for some time (and contributing to long-time correlations), and then escaping back to the main part of the stochastic region.

What we need is some way of bringing these particles back to the vicinity of the island. We do this by defining an $L \times L$ square around the island. This square spans the region $x_{\min} \leq x < x_{\max}$ and $-\frac{1}{2}L \leq y < \frac{1}{2}L$ where $L = x_{\max} - x_{\min}$. Whenever an orbit leaves this square at (x_t, y_t), we pick a new initial condition

$$(x_0, y_0) = (x_t - mL,\, y_t - nL)$$

with m and n being integers chosen so that (x_0, y_0) lies inside the square. We also record the length t of the previous orbit segment. This procedure defines the periodic quadratic map Q^*, which can be shown to sample the orbits close to the island in the same way that the general mapping does. Examples of the orbits of Q^* are shown in Fig. 2 for the same parameters as for Fig. 1.

One useful way of looking at Q^* is as a magnification of a small region near a tangent bifurcation in the general mapping. The difference is that once the trajectory leaves the vicinity of the islands, it is immediately re-injected on the other side of the islands. In the general map, the trajectory will spend some long time, which depends on the ratio of the size of the islands to the total accessible portion of phase space, in the stochastic sea before coming back to the vicinity of the islands.

Assuming that the long-time behavior of stochastic orbits is dominated by the region close to the islands, there are two advantages to reducing the problem to a study of Q^*. Firstly, since Q^* describes the behavior of most

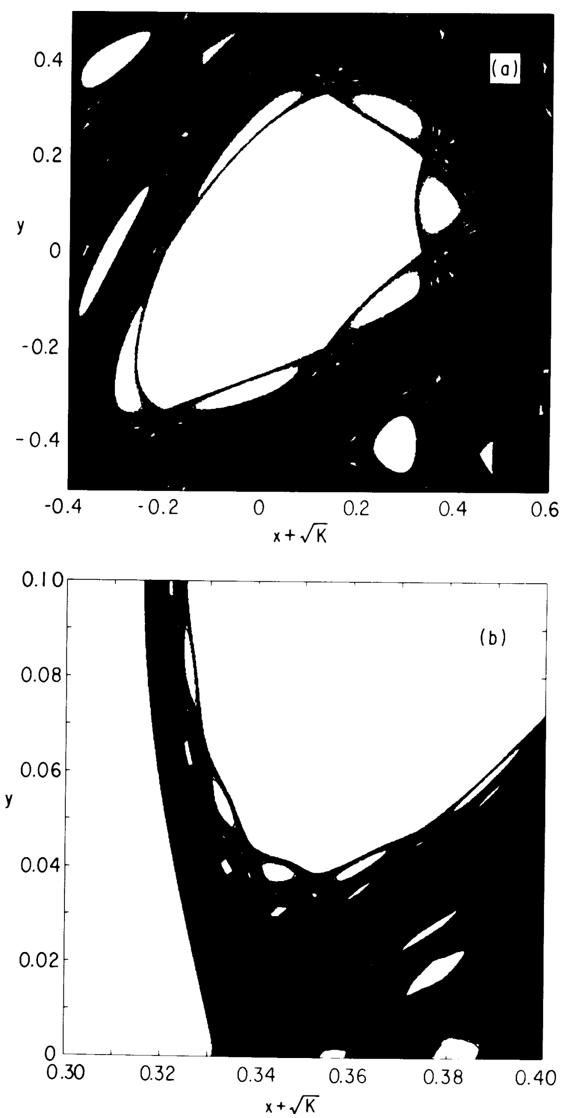

Fig. 2. (a) Stochastic trajectories for periodic quadratic map Q^* for $K = 0.1$. (b) An enlargement of a portion of (a). Here $x_{\min} + \sqrt{K} = -0.4$, $x_{\max} + \sqrt{K} = 0.6$.

islands far into the stochastic regime, the properties of many mappings may be treated by looking at a special mapping Q^* which depends only on a single parameter K. The second advantage is that the properties of orbits close to the islands may be studied much more efficiently because there is no need to follow orbits while they spend a long and uninteresting time far from the islands.

Trapping Statistics

The prescription for numerically determining the stickiness of the island system in Q is to compute a long orbit in the stochastic region of Q^*. The orbit is divided into segments at those points where it leaves the $L \times L$ square. The main results of the calculation are then the *trapping statistics* f_t which are proportional to the number of orbit segments which have a length of t. Suppose that the total length of the orbit is T and N_t is the number of segments of length t. If T is so large that we can ignore partial segments at the ends of the orbit, then we have $\sum tN_t = T$; the total number of segments is $N = \sum N_t$. The trapping statistics are defined by $f_t = N_t/T$ and are therefore normalized so that $\sum tf_t = 1$. The mean length of the orbits is given by $\alpha = 1/\sum f_t$ $(= T/N)$. The probability that a particular segment has length t is $p_t = \alpha f_t$ $(= N_t/N)$. If an arbitrary point is chosen in the orbit, then tf_t is the probability that this point belongs to a segment of length t and f_t is the probability that it belongs to the beginning, say, of a segment of length@t.

The survival probability

$$P_t = \sum_{\tau=t+1}^{\infty} p_\tau$$

is the probability that an orbit beginning in a segment at $t = 0$ is still trapped in the same segment at time t. Note that $P_0 = 1$ as required. This is the quantity studied by Channon and Lebowitz (1980) and Chirikov and Shepelyansky (1981). The correlation function

$$C_\tau = \sum_{t=\tau}^{\infty} (t-\tau) f_t = \sum_{t=\tau}^{\infty} P_t/\alpha$$

is the probability that a particle is trapped in the same segment at two times τ apart. Again, we have $C_0 = 1$.

There is another way of interpreting C_τ: Consider a drunkard who executes a one-dimensional random walk with velocity $v = dr/dt = \pm 1$. The direction of each step is chosen randomly, while the durations of the steps are chosen to be the lengths of consecutive trapped segments of Q^*. Then for integer τ, C_τ is just the usual correlation function for such a process, i.e., $\langle v_t v_{t+\tau} \rangle_t$. The behavior of this random-walk process is similar to the behavior of an orbit in the general mapping when two accelerator modes with opposite values of the acceleration are present. (This is the case with the first-order accelerator modes for the standard mapping.)

A diffusion coefficient may be defined by

$$D = \tfrac{1}{2} C_0 + \sum_{\tau=1}^{\infty} C_\tau = \sum \tfrac{1}{2} t^2 f_t.$$

This gives the diffusion rate for the drunkard in the random-walk problem above. It is also related to the diffusion coefficient for the general mapping.

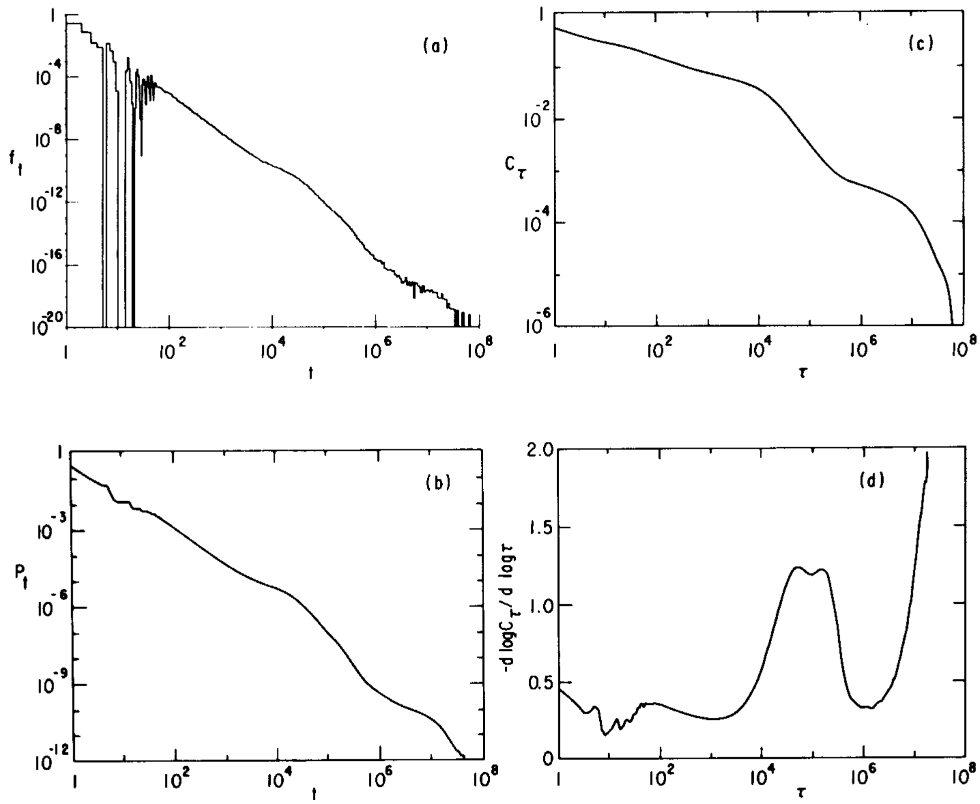

Fig. 3. (a) The trapping statistics f_t for $K = 0.1$. (b), (c), and (d) show P_t, C_τ, and $d \log C_\tau / d \log \tau$.

Results for $K = 0.1$

We have measured f_t for K between 0 and 1.3 at intervals of 0.05, and at most of the values of K a slow algebraic decay of f_t is seen. A representative case is $K = 0.1$, whose trapping statistics are given in Fig. 3(a), which illustrates the slow decay for very long times $t \sim 10^7$. Also given in Fig. 3 are P_t, C_τ, and $\alpha \equiv -d \log C_\tau / d \log \tau$ (thus locally $C_\tau \sim \tau^{-\alpha}$). This last plot shows the power at which C_τ decays varying between about $\frac{1}{4}$ and $\frac{3}{2}$.

A glance at Fig. 2 shows the origin of this behavior. The central island is surrounded by a chain of sixth-order islands. Around each of these islands are several other sets of islands. This picture repeats itself at deeper and deeper levels. A particle which manages to penetrate into this maze can get stuck in it for a long time.

For $\tau \lesssim 10^4$, Fig. 3(d) gives $\alpha \approx \frac{1}{4}$. Correspondingly we have $P_t \sim t^{-p}$ where $p = 1 + \alpha \approx 5/4$. This is close to the asymptotic ($t \to \infty$) result found by Chirikov and Shepelyansky (1981) for the whisker map, in which

$\langle p\rangle \approx 1.45$. However, in our case, α shows some strong variations beyond $\tau \approx 10^4$ where C_τ "steps down" (e.g., between 10^4 and 3×10^5). This means that the asymptotic form of C_τ is very difficult to determine numerically.

The diffusion coefficient D is given by the summation of C and is approximately 6400. The error in this estimate of D depends on the asymptotic form for C_τ. On the basis of the numerical results, we cannot rule out the possibility that as $\tau \to \infty$, C_τ decays with $\alpha \le 1$. In that case, D would be infinite!

If D is indeed infinite, we would wish to know how a group of particles spreads with time. We again consider the drunkard's walk based on Q^* which was introduced earlier. The second moment of r is related to the correlation function by

$$S_t \equiv \langle (r_t - r_0)^2\rangle = tC_0 + 2\sum_{\tau=1}^{t}(t-\tau)C_\tau.$$

This is plotted in Fig. 4(a), using the data of Fig. 3. For $t \lesssim 10^4$, S_t grows somewhat faster than $t^{3/2}$ (see Fig. 4(b)) and even until $t \approx 10^7$, S_t is growing significantly faster than linearly. Beyond 10^7, the numerical data shows a convergence to a linear rate; but this is merely because no segments longer than about 6×10^7 were observed. For $t\to\infty$, S_t grows as $t^{2-\alpha}$, assuming that the exponent α at which C_τ decays asymptotically is less than 1. If the diffusion coefficient is estimated from $D_t = \frac{1}{2}S_t/t$, then D_t grows with t as shown in Fig. 4(c).

Discussion

We can apply these results to the determination of the correlation function of a general mapping. Suppose the correlation function is defined by

$$\mathcal{C}(\tau) = \langle h(\mathbf{x}(t))h(\mathbf{x}(t+\tau))\rangle_t,$$

where h is some smooth function of the position in phase space $\mathbf{x}$. Then the contribution of an island located at $\mathbf{x}_0$ to $\mathcal{C}(\tau)$ is (Karney, 1983)

$$\mathcal{C}_{\rm is}(\tau) = h^2(\mathbf{x}_0)(B/A)C_\tau,$$

where A is the total area of the stochastic component of the general mapping and B is the portion of the that area which is near the island. (More precisely, when we regard the $L\times L$ square of Q^* as being a magnification of a small area of the general mapping, then B measures the area of the stochastic component within this small area.) Similar relations connect D and S_t and the corresponding quantities for the general mapping.

In the case of accelerator mode in the standard mapping, K is related to the parameter k by $k^2 = (2\pi n)^2 + 16K$ where n is an integer. For $K = 0.1$, we take 6400 as a lower bound for D. We find that the contribution to the diffusion coefficient is increased over its quasi-linear value by a factor of at

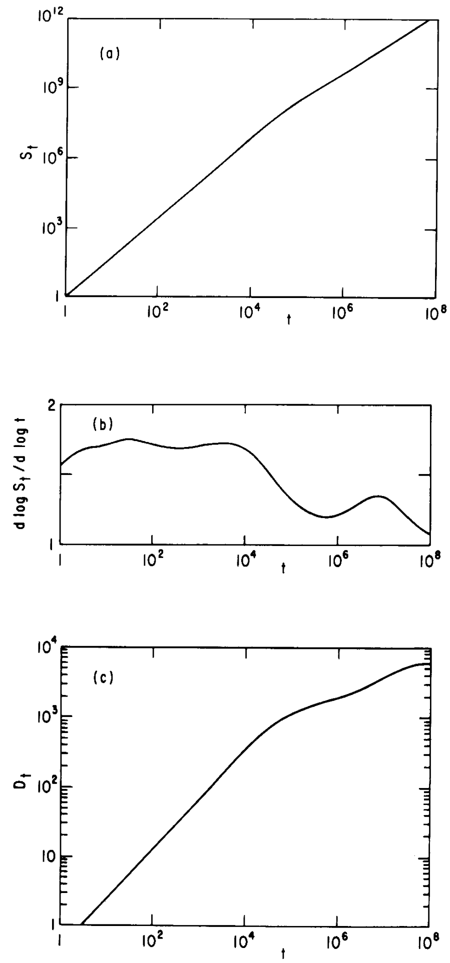

Fig. 4. (a) The variance S_t for the case given in Fig. 3. (b) and (c) show $d \log S_t / d \log t$ and $D_t = \frac{1}{2} S_t / t$.

least $360/n^2$. Thus for $n = 1$ or $k \approx 6.41$, the islands completely dominate the diffusion. The first-order accelerator modes continue to have such a large effect at least until $k \approx 100$. If D is in fact infinite, even arbitrarily small accelerator modes will eventually dominate the motion and Fig. 4 can be used to estimate the time at which the accelerator modes become important.

In summary, small islands within the stochastic sea lead to correlations in the stochastic orbits for extremely long times. When the islands are accelerator modes, this may cause the particles to behave non-diffusively, i.e., the mean squared displacement of the particles may increase faster than linearly with time.

In order to provide a definitive answer to this question, the asymptotic behavior of C_τ must be determined. Because the asymptotic regime starts at such a large τ (greater than 10^7), it appears its properties cannot be studied by the numerical method used in this paper. What is needed is a better analytical understanding of the behavior of trajectories close to the border between integrability and stochasticity. Chirikov (1982) has made some useful steps in this direction.

Acknowledgments

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