Solution of the three-wave resonant equations with one wave heavily damped

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The three wave equations in the limit where the waves at the upper two frequencies are undamped and the lowest frequency mode is heavily damped so that its dynamic equation becomes $\gamma_3 a_3 = K\mathring{a}_1 a_2^*$, are considered. These equations are solved (by quadrature) in two dimensions and time subject to arbitrary initial and boundary conditions. Illustrative examples arising in tokamak heating by lower hybrid waves are presented.

I. INTRODUCTION

The resonant interaction between three waves of action amplitudes a_j, $j=1, 2, 3$, whose frequencies and wave vectors obey the resonant conditions $\omega_1=\omega_2+\omega_3$, $\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3$ can be described by

$$a_{1t}+\mathbf{v}_1\cdot\nabla a_1+\gamma_1 a_1=-Ka_2a_3\,, \tag{1a}$$

$$a_{2t}+\mathbf{v}_2\cdot\nabla a_2+\gamma_2 a_2=K^*a_1a_3^*\,, \tag{1b}$$

$$a_{3t}+\mathbf{v}_3\cdot\nabla a_3+\gamma_3 a_3=K^*a_1a_2^*\,. \tag{1c}$$

In (1) the $\mathbf{v}$'s are the group velocities of the waves, the γ's are the damping constants, and K is the coupling coefficient. Equations (1) describe a variety of physical phenomena such as the nonlinear decay of lower hybrid waves in plasma[1] and laser-plasma interactions.[2] Various approximations of (1) have been studied both numerically and analytically.[3,4] Recently, Kaup[5] solved (1) by the inverse scattering method when $\gamma_j=0$, $j=1, 2, 3$. Here, we consider the equation when $\gamma_1=\gamma_2=0$ and the third mode is so heavily damped that (1c) becomes

$$\gamma_3 a_3=K^*a_1a_2^*\,. \tag{2}$$

In this limit, (1a, b) can be rewritten in terms of action densities as

$$I_{1t}+\mathbf{v}_1\cdot\nabla I_1=-I_1I_2\,, \tag{3a}$$

$$I_{2t}+\mathbf{v}_2\cdot\nabla I_2=I_1I_2\,, \tag{3b}$$

where $I_1=(2/\gamma_3)|Ka_1|^2$, $I_2=(2/\gamma_3)|Ka_2|^2$. The one-dimensional steady state solutions of these equations are readily obtained and have been thoroughly examined. In this paper we will show that the general initial or boundary value problem of (3) in two dimensions and time can be reduced to quadrature. Note that (3) can also be derived if the mismatch terms arising from inhomogeneities are included in (1a, b, c). Equations (3) have also been derived for the decay of lower hybrid waves into quasimodes.[6,7]

If (3) are one-dimensional, i.e., if $\nabla=\hat{x}(\partial/\partial x)$, the solutions of I_1 and I_2 have been found by Chu,[8] Chen,[9] and Hasimoto.[10] I_1 and I_2 are given as

$$I_1=-\log[Z(\xi)-T(\tau)]_\tau\,, \tag{4a}$$

$$I_2=\log[Z(\xi)-T(\tau)]_\xi\,, \tag{4b}$$

where $Z(\xi)$ and $T(\tau)$ are arbitrary functions of the independent variables

$$\xi=(x-v_{2x}t)/(v_{1x}-v_{2x})\,,\qquad \tau=(x-v_{1x}t)/(v_{2x}-v_{1x})\,. \tag{5}$$

In Sec. II, we will show how to find the functions $Z(\xi)$ and $T(\tau)$ for arbitrary initial conditions, and so solve the general initial value problem in one dimension. In Sec. III, we show how the results of Sec. II may be extended to solve initial value problems in higher dimensions. In Sec. IV, we will discuss the solution of the one-dimensional boundary value problem. In Sec. V, we will extend this boundary value problem to two spatial dimensions.

II. INITIAL VALUE PROBLEM IN ONE DIMENSION

We first present the initial value solution of the one-dimensional problem. If the initial values of I_1 and I_2 are $I_1(x,0)=P_1(x)$ and $I_2(x,0)=P_2(x)$, then we can solve for $Z[\xi(x,0)]=Z(x/V)$ and $T[\tau(x,0)]=T(-x/V)$ $[V=v_{1x}-v_{2x}]$. Equations (4) yield

$$P_1Z_x(x/V)+P_2T_x(-x/V)=0\,,$$

$$Z_x\left(\frac{x}{V}\right)-T_x\left(-\frac{x}{V}\right)=\frac{P_1(x)+P_2(x)}{V}\left[\exp\left(\int_0^x dz\,\frac{P_1(z)+P_2(z)}{V}\right)\right].$$

Then, $Z(\xi)$ and $T(\tau)$ are given by

$$Z(\xi)=\frac{1}{2}+\frac{1}{V}\int_0^{V\xi}dy\,P_2(y)\exp\left(\int_0^y dz\,\frac{P_1(z)+P_2(z)}{V}\right)\,, \tag{6a}$$

$$T(\tau)=-\frac{1}{2}-\frac{1}{V}\int_0^{-V\tau}dy\,P_1(y)\exp\left(\int_0^y dz\,\frac{P_1(z)+P_2(z)}{V}\right)\,. \tag{6b}$$

The analytic solutions for I_1 and I_2 are

$$I_1=-\left(v_{2x}\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\log\left[Z\left(\frac{x-v_{2x}t}{V}\right)-T\left(\frac{x-v_{1x}t}{-V}\right)\right], \tag{7a}$$

$$I_2=\left(v_{1x}\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\log\left[Z\left(\frac{x-v_{2x}t}{V}\right)-T\left(\frac{x-v_{1x}t}{-V}\right)\right]. \tag{7b}$$

As an example, consider the collision of two Gaussian wave packets:

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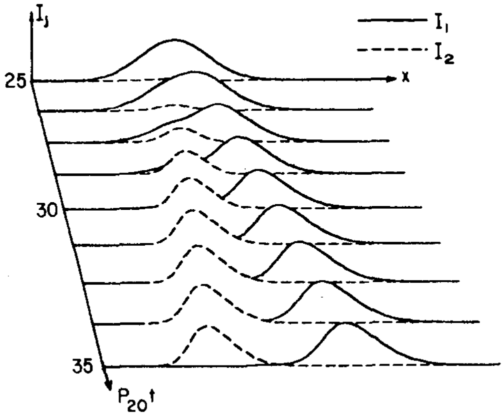

FIG. 1. Plot of $I_1(x,t)$ and $I_2(x,t)$ for the one-dimensional initial value problem, when $v_{2x}=-1.2v_{1x}$, $P_{20}=10^{-3}P_{10}$, $w_1=10\ v_{1x}/P_{10}$, $w_2=w_1/2$, $x_1+x_2=60\ v_{1x}/P_{10}$.

$$P_1(x)=2\pi^{-1/2}P_{10}\exp\left[-\left(\frac{x+x_1}{w_1}\right)^2\right],$$

$$P_2(x)=2\pi^{-1/2}P_{20}\exp\left[-\left(\frac{x-x_2}{w_2}\right)^2\right].$$

We take $v_{1x}>v_{2x}$. If x_1+x_2 is sufficiently large, P_1 and P_2 can be considered non-overlapping and (6) may be evaluated to give

$$Z(\xi)=-\tfrac{1}{2}+\exp\left\{\frac{P_{20}w_2}{V}\left[\operatorname{erf}\left(\frac{V\xi-x_2}{w_2}\right)+1\right]\right\},$$

$$T(\tau)=\tfrac{1}{2}-\exp\left\{\frac{P_{10}w_1}{V}\left[\operatorname{erf}\left(\frac{-V\tau+x_1}{w_1}\right)-1\right]\right\}.$$

An example of this interaction is plotted in Fig. 1. Note that for wave 2, the leading edge is amplified more than its trailing edge, since it sees an undepleted wave 1. This effect is generally true for all initial pulses.

From the general expressions (6) for Z and T, we can also calculate the final actions of modes 1 and 2 (i.e., the areas of I_1 and I_2) if initially the pulses are non-overlapping

$$A_{1f}+A_{2f}=A_{1i}+A_{2i}, \quad (8a)$$

$$A_{2f}=V\log[\exp(A_{2i}/V)+\exp(-A_{1i}/V)-1]+A_{1i}. \quad (8b)$$

Here, A_{jf} and A_{ji} are the final and initial actions of mode j. Equation (8a) is the equation for conservation of action. Equation (8b) gives the transfer of action from wave 1 to wave 2 and enables us to calculate the energy dissipated in mode 3. Note that the energy dissipated is independent of the initial shapes of the waves.

III. INITIAL VALUE PROBLEM IN TWO DIMENSIONS

The solution to the two-dimensional initial value problem is easily reduced to the one-dimensional problem by a coordinate transformation. For example, we may effect this by a Galilean transformation to a coordinate system in which $\mathbf{v}_1$ and $\mathbf{v}_2$ are collinear followed by a rotation such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel to $\hat{\mathbf{x}}$. In this coordinate system $v_{1y}=v_{2y}=0$, and the equations reduce to the one-dimensional ones. An alternative transformation is given in connection with the boundary value problem in Sec. V.

IV. BOUNDARY VALUE PROBLEM IN ONE SPATIAL DIMENSION

There are two boundary value problems that can be solved. If $v_{1x}>0$ and $v_{2x}>0$, then (1), with the conditions $I_1(0,t)=Q_1(t)$, $I_2(0,t)=Q_2(t)$, $I_1(x,0)=P_1(x)$, $I_2(x,0)=P_2(x)$, $0<x<L$ can be solved in the same manner as the initial value problem because I_1 and I_2 are specified on the same lines in x, t space. Moreover, if $v_{1x}>0$ but $v_{2x}<0$, then the boundary values for (1) become

$$I_1(0,t)=Q_1(t), \quad I_2(L,t)=Q_2(t), \quad (9a)$$

$$I_1(x,0)=P_1(x), \quad I_2(x,0)=P_2(x). \quad (9b)$$

By using the method described in solving the initial value problem, (6) will give the solution of

$$Z(\xi)=Z_0(x,V), \quad 0<x<L, \quad 0<V\xi<L,$$

$$T(\tau)=T_0(-x/V), \quad 0<x<-L, \quad -L<V\tau<0.$$

At the boundary at $x=0$, for $0<t<-L/v_{2x}$, $Z=Z_0(\xi)$ is known. We can solve for $T(\tau)=T_1(\tau)$, for $0<\tau<-v_{1x}L/v_{2x}V$, since from (4a)

$$T_{1\tau}=Q_1(V\tau/v_{1x})[Z_0(-v_{2x}\tau/v_{1x})-T_1(\tau)].$$

This procedure can be repeated successively at $x=0$ and $x=L$ to give the functions $Z_n(\xi)$ and $T_n(\tau)$, where

$$Z(\xi)=Z_n(\xi) \quad \text{for } \xi_{n-1}<\xi<\xi_n,$$

$$\xi_n=(L-v_{2x}\tau_{n-1})/v_{1x},$$

and

$$T(\tau)=T_n(\tau) \quad \text{for } \tau_{n-1}<\tau<\tau_n,$$

$$\tau_n=-v_{1x}\xi_{n-1}/v_{2x},$$

with $\xi_0=L/V$, and $\tau_0=0$. The relationships of τ_n and ξ_n are illustrated in Fig. 2. The functions Z_n and T_n can be solved for recursively to give

$$Z_n(\xi)=\mathfrak{Z}_n Z_H(\xi)-Z_H(\xi)$$
$$\times\int^{\xi}\left\{\left[Q_2\left(\frac{L-V\xi}{v_{2x}}\right)T_{n-1}\left(\frac{L-v_{1x}\xi}{v_{2x}}\right)\right]Z_H^{-1}(\xi)\right\}d\xi, \quad (10a)$$

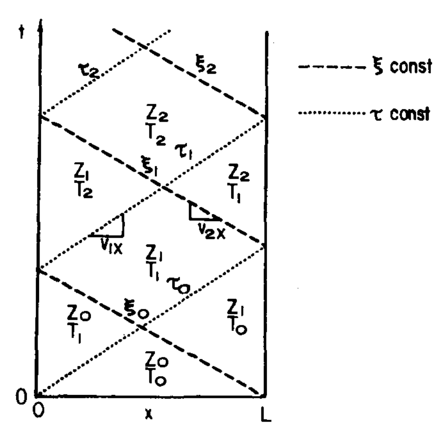

FIG. 2. Plot of ξ_n and τ_n in the x-t plane.

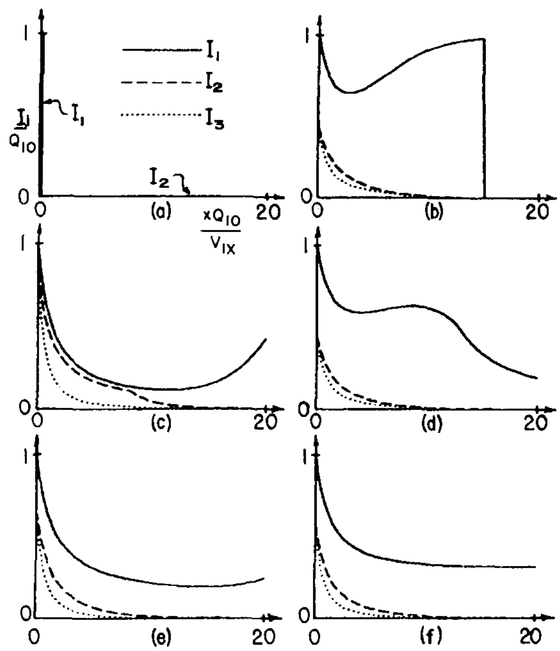

FIG. 3. Plots of $I_1(x,t)$, $I_2(x,t)$, and $I_3(x,t)=4\,|Ka_3|^2\,Q_{10}$ for the one-dimensional boundary value problem when $v_{2x}=-1.2v_{1x}$, $L=20v_{1x}/Q_{10}$, $Q_{20}=10^{-3}Q_{10}$, and for various times t. (a) 0, (b) $15/Q_{10}$, (c) $30/Q_{10}$, (d) $50/Q_{10}$, (e) $70/Q_{10}$, (f) steady state.

$$T_n(\tau)=\mathfrak{T}_n T_H(\tau)+T_H(\tau)\int^{\tau}\left\{\left[Q_1\left(\frac{V\tau}{v_{1x}}\right)Z_{n-1}\left(\frac{-v_{2x}\tau}{v_{1x}}\right)\right]T_H^{-1}(\tau)\right\}d\tau\ , \tag{10b}$$

where $Z_H(\tau)=\exp\{\int^{\xi}Q_2[(L-V\xi)/v_{2x}]d\xi\}$ and $T_H(\tau)=\exp[\int^{\tau}-Q_1(V\tau/v_{1x})d\tau]$. The constants of integration $\mathfrak{Z}_n$ and $\mathfrak{T}_n$ are determined by requiring the continuity of $T(\tau)$ and $Z(\xi)$. For simple values of the initial and boundary conditions, Z_n and T_n and hence $I_1(x,t)$ and $I_2(x,t)$ can be solved analytically. For example, if

$$Q_1(t)=Q_{10}\,(\text{const})\ ,\qquad P_1(x)=0\ ,$$

$$Q_2(t)=Q_{20}\,(\text{const})\ ,\qquad P_2(x)=Q_{20}\ ,$$

this boundary value problem describes the growth of wave 2 from noise when wave 1 is turned on at $t=0$. Figure 3 illustrates this example. Note that the amplitude of wave 2 oscillates considerably. In Fig. 4, $I_1(L,t)$ and $I_2(0,t)$ are plotted. Both waves oscillate but decay with time until they reach the steady state shown in Fig. 3(f).

V. BOUNDARY VALUE PROBLEM IN TWO DIMENSIONS

By the same procedure that the two-dimensional initial value problem reduces to the one-dimensional problem, we can reduce the two-dimensional boundary value problem to a one-dimensional problem. However, this will generally result in moving boundaries, and while this presents no intrinsic difficulty, we can often solve a simpler problem (in which some or all of the boundaries are stationary) by using a shear transformation. This is best illustrated by the example of the decay of lower hybrid waves in tokamaks. Using a slab geometry, we will assume that the pump (I_1) is excited by wave guides at $x=0$. Initially, the second lower hybrid wave is noise. If $v_{2x}<0$, the initial and boundary conditions are

$$I_1(x=0,y,t)=Q_1(y,t)\ , \tag{11a}$$

$$I_2(L,y,t)=Q_2(y,t)\ . \tag{11b}$$

Using the coordinate transformation

$$t'=t\ ,\qquad x'=x\ ,\qquad y'=y-ut-\alpha x\ , \tag{12a}$$

$$\alpha=\frac{v_{1y}-v_{2y}}{v_{1x}-v_{2x}}\ ,\qquad u=\frac{v_{1x}v_{2y}-v_{2x}v_{1y}}{v_{1x}-v_{2x}} \tag{12b}$$

(3) becomes

$$I_{1t'}+v_{1x}I_{1x'}=-I_1I_2\ , \tag{13a}$$

$$I_{2t'}+v_{2x}I_{2x'}=I_1I_2\ , \tag{13b}$$

in which y' is a parameter. [Note that the transformation (12) may also be used to reduce the initial value problem in two dimensions to the one-dimensional problem (see Sec. III).] The transformed boundary conditions are

$$I_1(x',y',t')\equiv Q_1'(y',t')=Q_1(y'+ut',t')\ , \tag{14a}$$

$$I_2(x'=L,y',t')\equiv Q_2'(y',t')=Q_2(y'+ut'+L',t')\ . \tag{14b}$$

Since y' is a parameter in (13) for a given y', this is again reduced to a one-dimensional boundary value problem.

For example, if we assume that Q_1 is excited by a waveguide of width l (Fig. 5), so that

$$Q_1(y,t)=\begin{cases}Q_{10}(\text{const})\ , & 0<y<l\ ,\quad t>0\\ 0 & \text{otherwise}\end{cases}\ , \tag{15a}$$

and Q_2 is initially noise

$$Q_2(y,t)=Q_{20}(\text{const})\ , \tag{15b}$$

then for a given y'

$$Q_1(y'+ut',t')=\begin{cases}Q_{10}\ , & -y'<ut'<l-y'\ \text{ and }\ t'>0\\ 0\ , & \text{otherwise}\end{cases}$$

$$Q_2(y'+ut'+L,t')=Q_{20}\ .$$

The solution of this can be found exactly and this is shown in Fig. 6. We have taken $l/u<L(v_{1x}^{-1}+|v_{2x}|^{-1})$ in this figure, so that the width of the pump is less than the "bounce width." (This is typical of problems involving the decay of a lower hybrid pump.) Note that the noise level has been chosen unrealistically large, in order to illustrate the interaction. This was necessary because

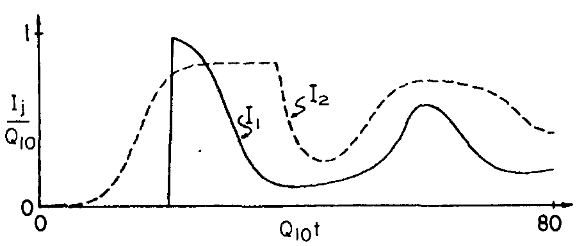

FIG. 4. Plot of $I_1(L,t)$ and $I_2(0,t)$ for the one-dimensional boundary value problem.

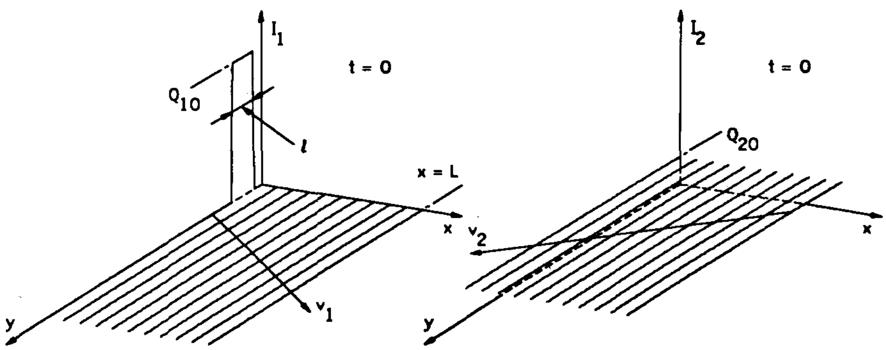

FIG. 5. Initial conditions for the two-dimensional boundary value problem. Here $v_{2x} = -1.2\, v_{1x}$, $v_{2y} = 1.4\, v_{1y}$, $Q_{20} = 0.1\, Q_{10}$, $L = 20\, v_{1x}/Q_{10}$, $f = 5\, v_{1y}/Q_{10}$.

with this geometry convective losses due to the finite width in y minimize the interaction. This should be compared with Fig. 3 (for which $l \to \infty$) which has the same interaction length L, but a much smaller noise level. Clearly, the depletion of the pump is maximized in the two-dimensional case by choosing $\mathbf{v}_1$ and $\mathbf{v}_2$ to be as close to being parallel as possible (subject to the resonance conditions, and assuming a weak dependence of the coupling coefficient K on the geometry). Note that the interaction in Fig. 6 reaches a steady state in finite time; the pump reaches its steady state condition in one bounce time

$$t = L/v_{1x} + l/u \,, \tag{16a}$$

while wave 2 reaches its steady state in two bounce times

$$t = L(1/v_{1x} + 1/|v_{2x}|) + l/u \,. \tag{16b}$$

VI. CONCLUSIONS

We have shown that initial value and boundary value problems can be solved exactly in one and two dimensions. This solution technique can easily be extended to higher dimensions. An accurate evaluation of the nonlinear stages of several important parametric or quasi-mode interactions is now possible. By carefully examining the times and regions of interest, the dimensionality of the problem can often be reduced. For example, the problem of the decay of lower hybrid waves, discussed in Sec. V, is a problem in two spatial dimensions and time. However, the time that the lower hybrid pump takes to reach steady state (16a), is typically much shorter than the duration of the pulse of rf power. Thus, for most purposes the problem reduces to the solution of the two-dimensional steady state problem. This is always reducible to an equivalent problem in one dimension and time, and so is soluble by the methods of Secs. II and IV. For example, if $v_{1x} > 0$ and $v_{2x} > 0$, x may be considered as a time-like variable and the equations may be put in the one-dimensional form of (3) by letting $I_1 = S_1/v_{2x}$ and $I_2 = S_2/v_{1x}$. Of course, the solution will be effectively one-dimensional if the pump is depleted by one transit of the noise, I_2, through it. This is usually not the case, since lower hybrid waves are characterized by large group velocities parallel to the magnetic field. In the two-dimensional case, if the initial noise level, Q_{20}, is small, we can estimate the total depletion of I_1 as a function of x by means of (8). To compute the power flow in I_1 at $x = x_0$ we take the length of the noise "pulse" at $x = 0$ to be $w_2 = x|v_{1y}/v_{1x} - v_{2y}/v_{1x}|$. The error arises from neglecting the effect of overlap at $x = 0$ and $x = x_0$, and so the approximation is valid in the limit where the width of the waveguides, $l \ll w_2$. It is easy to check this result against the exact results, if the need arises. Chen and Berger[6] have applied our results to the problem of lower hybrid heating in a tokamak. They considered the decay of a lower hybrid pump (ω_1) into a lower hybrid sideband (ω_2) and a quasi-mode (ω_3). Their results indicate that this process can cause significant pump depletion near the plasma edge, particularly for a reactor sized toka-

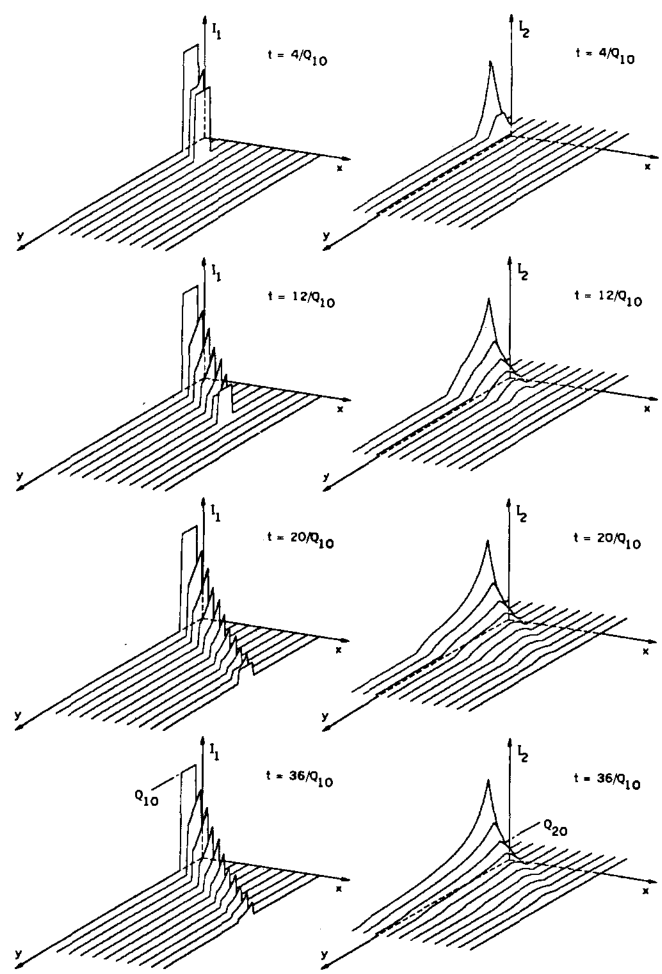

FIG. 6. Temporal development of the two-dimensional boundary value problem with initial conditions as given in Fig. 5.

mak. A fraction, ω_3/ω_1, of the input power is deposited into the quasi-mode leading to surface heating. However, a thorough examination of this process for various plasma and machine parameters is needed before the role of quasi-mode decay in lower hybrid heating can be assessed.

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