Effect of resonance broadening on the evolution of the edge of a turbulent spectrum

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The extent to which nonlinear wave-particle resonance broadening results in a narrowing of an incident lower-hybrid wave spectrum is investigated. This narrowing is of concern because it could make control of lower-hybrid heating difficult. It is shown numerically, however, that relatively uniform spatial power deposition occurs if resonance broadening effects are treated consistently on both the wave spectrum and the particle distribution.

I. INTRODUCTION

The injection of lower hybrid waves into a tokamak plasma has been suggested as a means of heating the plasma to ignition[1] or as a means of driving the torodial plasma current.[2] The success of both schemes is based on the absorbtion of the rf power by plasma electrons and depends, in part, on the expectation that the spatial deposition of the rf power can be controlled. In particular, for heating electrons or generating current, it is likely that the most desirable deposition of lower hybrid rf power would be near the plasma center where the temperature and density profiles are relatively flat. The goal of the present study is to assess whether the nonlinear effect of resonance broadening[3] interferes with the control that we hope to retain over the power depostion.

To facilitate our assessment of resonance broadening effects, we must model the interaction of the waves with the plasma in a manner that is realistic, yet isolates the effects of resonance broadening from other effects. Specifically, we shall model the plasma as a homogeneous slab. We are motivated by the following qualitative description of the spatial deposition of the rf power: We imagine that the parallel (to the magnetic field) wave phase velocity is chosen large compared with the electron thermal velocity near the plasma periphery, but only three to four times the thermal velocity at some hotter interior point. Thus, exponentially few electrons are resonant with the wave near the cool periphery so that the wave propagates relatively undamped until it reaches an interior point. There, the plasma is warmer so that a substantial number of resonant electrons are present. The wave power is absorbed between this interior point and the plasma center (the magnetic axis). To the extent that the interior point in question and the plasma center are close together, the plasma can be modeled as nearly homogeneous. Although the model may not be entirely accurate, it does isolate the resonance broadening effects from the effects of temperature and density gradients. The other simplification that we employ in modeling the power deposition is that the waves are incident upon an infinite half-space, rather than converging upon the magnetic axis. This modeling isolates the resonance broadening effects from the effects of cylindrical geometry.

In order to understand the role of resonance broadening with regard to the spatial deposition of rf power, it is important to distinguish low-power injection from high-power injection. By low-power injection we mean that the rf power is so weak, or the plasma collisionality is so strong, that the electron velocity distribution remains nearly Maxwellian. (It is understood that the relative plasma collisionality depends, in part, on the wave phase velocity.[2]) In this limit the damping coefficient for the waves is independent of their power. In a uniform plasma, the transmitted wave power would decay exponentially with distance into the plasma since the damping coefficient is proportional to the velocity derivative of the distribution function. That is, $dD/dt = \gamma D$, where D is a measure of the incident rf power and where $\gamma \propto \partial F/\partial v$ is essentially independent of D.

In the opposite limit of high-power rf waves, the collisionality of the plasma is too small to restore the Maxwellian electron velocity distribution. Instead, a plateau forms on the tail of the parallel velocity distribution at the velocity corresponding to the wave phase velocity. Thus, in the high-power limit, the slope of the distribution function and, consequently, the damping coefficient for the waves is inversely proportional to the wave power so that it is the rate of power deposition, and not the damping coefficient, that is independent of the wave power, i.e., $dD/dt \propto (1/D)D$. Therefore, the transmitted wave power would only decay linearly with distance into a uniform plasma. However, even when high-power lower hybrid waves are injected into the plasma, there are some components in the velocity-space spectrum, in particular, those near the edge of the spectrum, whose decay is governed by the low-power limit. That there exists some spectral components in the low-power regime is assured if the spectrum is continuous. Thus, we expect that under high-power injection, the edge of the wave spectrum must sharpen because the intensity of the components near the spectrum edge is in the low-power regime and they experience exponential decay relative to the linear decay of the components in the central portion of the spectrum.

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Associated with this sharpening of the spectrum, there would be a narrowing of the spectrum because of the faster decay of the edge of the spectrum. However, narrowing can occur independently, i.e., even when the spectrum is flat. The narrowing of the spectrum can occur by virtue of resonance broadening, which assures, among other things, that the damping rate of the spectral components undergoes a continous transition between low- and high-power limits. In particular, the spectral components within a resonance broadening width[3] of the spectrum edge would experience a damping rate larger than the components in the center of the spectrum. Physically, this corresponds to these spectral components exchanging energy with electrons that have velocities outside the range of the wave phase velocities associated with the spectrum, i.e., with electrons for which the velocity distribution function presumably has a larger slope. If the resonance broadening width is broader than the characteristic width associated with the edge of the wave spectrum, then it can be imagined how the spectrum can narrow independently of the sharpening of the edge.

Even a small narrowing of the wave spectrum may be of great importance because it affects the value of v_1, the lowest parallel phase velocity in the wave spectrum. The number of resonant electrons in the steady state scales as $N_r \sim \exp(-v_1/2v_{te}^2)$, where v_{te} is the electron thermal velocity and is a function of distance into the plasma. The sensitive dependence of N_r on v_1 is reflected in the power deposition. A small narrowing of the spectrum implies that the power in the edge spectral components is quickly (exponentially) absorbed by the plasma. As a consequence, v_1 becomes a function of distance into the plasma. Associated with the increase in v_1, the plasma becomes transparent to the central spectral components. Thus, the narrowing of the spectrum would give a very unwelcome profile of power deposition.

The concern of this study is whether, due to resonance broadening, the narrowing of the wave spectrum occurs in the high-power limit. This concern is particularly appropriate in the case of current generation in tokamak reactors where one starts with a narrow spectrum incident on the plasma. We will, in fact, show that this narrowing does not occur when the effects of resonance broadening are included in a proper and self-consistent manner in the evolution of both the wave spectrum and the electron velocity distribution. The concern, although eventually discounted, is nevertheless genuine and could not have been alleviated without a numerical calculation. We will show how a more naive formulation of the problem, including the resonance broadening effects on the evolution of the wave spectrum only, does lead to a narrowing of the spectrum and an unfavorable deposition profile in the manner that has been described.

The paper is organized as follows: In Sec. II we write down the basic equations and pose the concern raised in this section in a more quantitative manner. The derivation of these equations is found in the Appendix. In Sec. III we present the numerical solution of the resonance broadening equations and demonstrate the importance of self-consistency in formulating these equations. We conclude with a discussion of our results in Sec. IV.

II. BASIC EQUATIONS

The evolution of the parallel electron velocity distribution F may be described by a Fokker–Planck equation (written in parallel velocity only) with an added quasi-linear diffusion term due to the waves, i.e.,

$$\frac{\partial F}{\partial \tau} = \frac{\partial}{\partial w}\langle D(w,\xi,\tau)\rangle \frac{\partial F}{\partial w} + \left(\frac{\partial F}{\partial \tau}\right)_c , \qquad (1)$$

where the quasi-linear diffusion coefficient D depends on F[see Eqs. (4) and (5)]. All quantities are written using the normalizations introduced in the Appendix. Thus, τ is time in units of inverse collision frequency, w is parallel electron velocity in units of the electron thermal speed, and ξ, the normalized spatial variable, is a measure of distance into the plasma. We use the averaging operator $\langle\ \rangle$ defined by

$$\langle G(w)\rangle \equiv \frac{1}{2\delta}\int_{w-\delta}^{w+\delta} G(w')dw' , \qquad (2)$$

where δ is a given resonance broadening width and $G(w)$ is an arbitrary function, which is averaged over this width. The collision operator is given by

$$\left(\frac{\partial F}{\partial \tau}\right)_c = \frac{\partial}{\partial w}\frac{1}{w^3}\frac{\partial F}{\partial w} + \frac{1}{w^2}F . \qquad (3)$$

This operator is linearized and written in the high-velocity limit. Also, a Maxwellian distribution has been assumed for the perpendicular velocity direction. The justification for using this collision operator in the present problem follows the arguments offered in Refs. 2 and 4. The wave diffusion coefficient D, which is proportional to the incident wave power (see the Appendix), evolves according to the equation

$$\frac{\partial D}{\partial \tau} + V_g \frac{\partial D}{\partial \xi} = \gamma(w,\xi,\tau)D , \qquad (4)$$

where $V_g = w$ and ξ are the dimensionless radial group velocity and spatial coordinate as shown in the Appendix. The wave damping rate γ is given by

$$\gamma = Aw|w|\frac{\partial\langle F\rangle}{\partial w} , \qquad (5)$$

where the constant A is defined in Eq. (A12). We shall discuss some of the properties of Eqs. (1)–(5).

Resonance broadening allows waves to exchange momentum with particles traveling at not exactly the wave parallel phase velocity, i.e., such that $\omega/k_\parallel \neq v_\parallel$. Since the ratio of wave momentum to wave energy is not the same as the ratio of particle momentum to particle energy for $\omega/k_\parallel \neq v_\parallel$, it follows immediately that if momentum is conserved, then energy cannot be conserved in this simplistic model of the interaction in one dimension. More sophisticated treatments are required to assure energy conservation, but the advantages of these sophisticated treatments are not necessary for the present purposes. Note that if collisions are neglected, i.e., $(\partial F/\partial \tau)_c = 0$, Eqs. (1)–(5) conserve energy only if the resonance broadening width δ is equal to zero.

(Note that momentum is conserved even for $\delta \neq 0$.) When δ is not equal to zero, conservation of energy is maintained only if we do not use a function to approximate the resonance broadening operator derived in Ref. 3. However, the equations would then assume a far more complicated form. Alternatively, we could force energy conservation in the manner described in Ref. 5, but we do not consider this necessary for the present application. Energy is nearly conserved if δ is small. Moreover, in the presence of collisions, the energy and momentum of the resonant electrons and waves are not separately conserved. Thus, the solution for F in the presence of collisions is not sensitive to small discrepancies in the separate balance of energy and momentum between the resonant electrons and the waves.[6] On the other hand, Eqs. (1)–(5) do advantageously preserve the non-negative nature of both F and D, no matter how δ is chosen. Thus, for the present application, where the energy that the resonant electrons gain from the waves is to be balanced against the energy they lose by colliding with nonresonant electrons, our approach of introducing the phenomena of resonance broadening in an approximate but simple manner (with desirable mathematical properties) retains the essential physics.

Our interest lies in obtaining the steady-state solution of Eqs. (1)–(5). Taking $\partial F/\partial \tau = 0$ in Eq. (1), we immediately find[2]

$$F = \exp\left(-\int_c^w \frac{w}{1+\langle D\rangle w^3} dw\right) , \tag{6}$$

where $c(\xi)$ is determined by the condition that the electron density [i.e., $F(w)$ integrated over the parallel velocity w] remains a (given) function of ξ only. Note that $F(w)$ is Maxwellian where $\langle D\rangle$ vanishes and is flat where $\langle D\rangle$ is large. Furthermore, note that the height of the plateau where F is flat is exponentially sensitive to the value of the slowest phase velocity in the wave spectrum.

The power carried by the transmitted wave may now be determined using Eqs. (4)–(6) with $\partial D/\partial \tau = 0$. The concern regarding resonance broadening, expressed in Sec. I, stems from Eq. (4). Near the spectrum edge, the damping rate tends to be much larger when the averaged F is employed instead of the unaveraged F. The effect of this larger damping rate is that the edge spectral components are lost faster than the central components. However, we shall see that this effect is mitigated when the D used in Eq. (6) is averaged over the resonance broadening width, as opposed to not being averaged. The reduction of the resonance broadening effect occurs because the particle distribution becomes flattened somewhat even outside the range of the spectrum phase velocities when $\langle D\rangle$ is used.

In the next section, we will present numerical solutions of Eqs. (4)–(6). When the resonance broadening effects are included in both evolution equations, i.e., for the waves and for the electrons, we refer to the solutions of the consistent set of equations. What we refer to as solutions of the inconsistent set of equations are solutions of Eqs. (4)–(6) but with the unaveraged value of D (naively) employed in Eq. (6) instead of the resonance broadened value $\langle D\rangle$. It will be shown that it is only the inconsistent set of equations that exhibits the severe narrowing of the spectrum and the consequent unfavorable power deposition profile.

III. NUMERICAL SOLUTION

In order to show the effect of resonance broadening on the propagation of lower hybrid waves and deposition of their power, we choose a typical set of parameters and display various aspects of the steady-state solutions of Eqs. (4)–(6) with (a) no resonance broadening, i.e., $\delta = 0$; (b) consistent resonance broadening; and (c) inconsistent resonance broadening, i.e., $\delta \neq 0$ in Eq. (5) and $\delta = 0$ in Eq. (6).

The method of solution involves starting at $\xi = 0$ where D is specified on a uniform grid of spacing Δw. $F(w, \xi = 0)$ is determined from Eq. (6) by numerical quadrature (using the rectangle rule). This is numerically differenced to give γ from Eq. (5) which is substituted into Eq. (4). The wave spectrum is then determined at $\xi = \Delta\xi$ by solving Eq. (4) using Euler's method. The resonance broadening operator, Eq. (2), is numerically evaluated using the rectangle rule. We set $\Delta w = 10^{-2}$ and $\Delta\xi = 10^{-3}$.

We chose $D(w, \xi = 0) = 30/w$ for $3.6 \leq w \leq 6.0$ and zero outside this range of parallel phase velocity. This is representative of lower hybrid waves with frequency of 1.2 GHz and power levels of 1 MW propagating in a plasma with electron density 2.5×10^{13} cm^{-3} and electron temperature $T_e = 2.5$ keV. The resonance broadening width is taken to be constant equal to $0.1\, v_{te}$, i.e., $\delta = 0.1$. Taking δ to be a constant and making no attempt to relate it back to $(D/k_{\parallel})^{1/3}$ is not strictly correct but suffices here in our examination of the nature of the influence that resonance broadening has on the deposition of power.

The variation of the spectral power density as a function of ξ is shown in Fig. 1. It is seen that with no resonance broadening, curve (a), and consistent resonance broadening, curve (b), the decay of the spectrum is nearly linear. The decay in case (b) is somewhat faster than in case (a). However, with inconsistent resonance broadening, curve (c), a rapid initial decay of the spectrum is followed by a much slower decay of the spectrum. Case (c) is illustrative of the effect that we had feared before doing the problem self-consistently.

The results shown in Fig. 1 are illustrated more succinctly in Figs. 2 and 3. In Fig. 2 we plot, as a function of ξ, the fraction of the total power carried by the lower hybrid waves and in Fig. 3, the rate of power deposition. Note in Fig. 3 the uniform deposition of power in case (a) and (b) compared with the nonuniform deposition in case (c). The origin of this behavior can be seen in Fig. 4 where we plot the location of the inner (low velocity) edge of the spectrum as a function of ξ for the three cases. The location of this edge is defined by the lesser of the two solutions to

$$D(w_1) = 0.5\ \max_w[D(w)] , \tag{7}$$

whereas, in cases (a) and (b), the edge is nearly station-

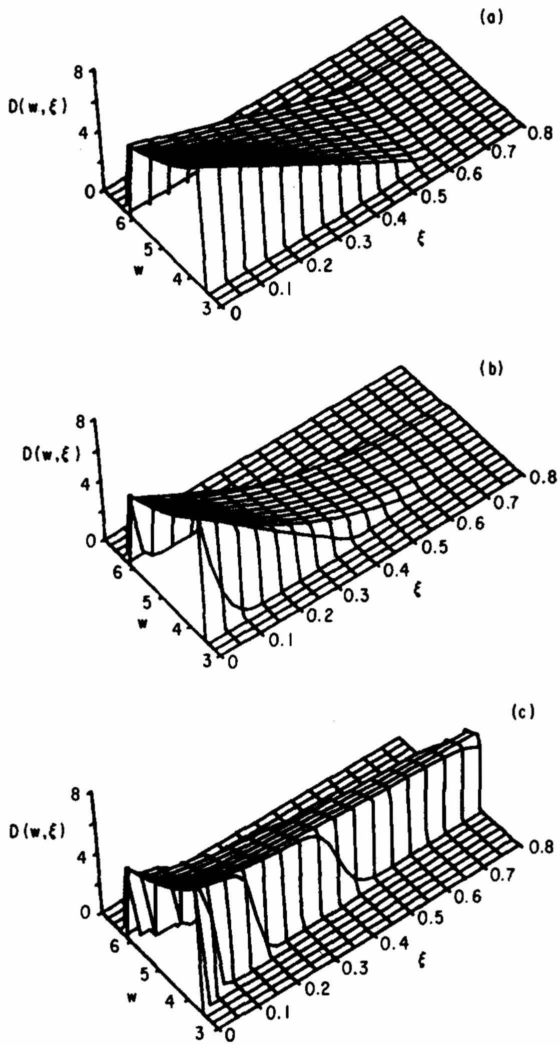

FIG. 1. Wave power spectrum (in parallel velocity space) as a function of depth into the plasma. At the $\xi = 0$, the power is zero except when $3.6 < w < 6.0$. In case (a) the effects of resonance broadening are omitted; in case (b) resonance broadening is included; and in case (c) resonance broadening is partially taken into account and thus treated inconsistently.

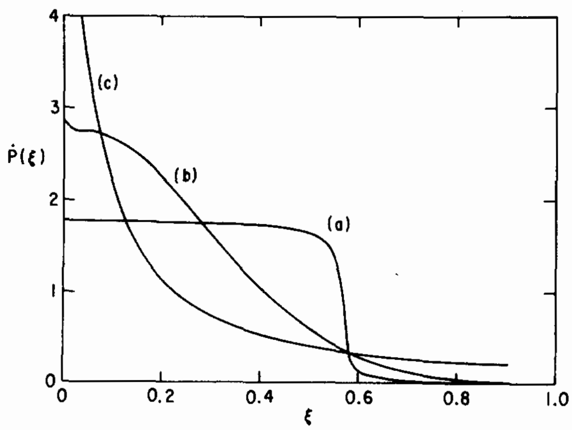

FIG. 3. Rate of rf-power deposition as function of depth into the plasma; (a) resonance broadening omitted, (b) resonance broadening included and (c) resonance broadening partially taken into account.

ary until a major fraction of the rf energy has been deposited, in case (c) the edge moves to larger velocities with increasing ξ (propagation into the plasma), reducing the number of resonant particles and hence the damping rate.

IV. CONCLUSIONS

We have examined the extent to which nonlinear resonance broadening affects a lower hybrid rf spectrum. While one might expect resonance broadening on the waves to dramatically narrow the spectrum, it is shown that this effect is counteracted by resonance broadening on the particles, which extends the plateau in the parallel velocity distribution a resonance broadening width into the nonresonance region. Thus, when resonance broadening is treated consistently, the rf power spectrum does not narrow significantly and the uniform spatial deposition of the rf power is retained. We have considered the effects of resonance broadening on a rectangular incident wave spectrum. Resonance broadening has a greater effect on this power spectrum than on a more realistic wave power spectrum.

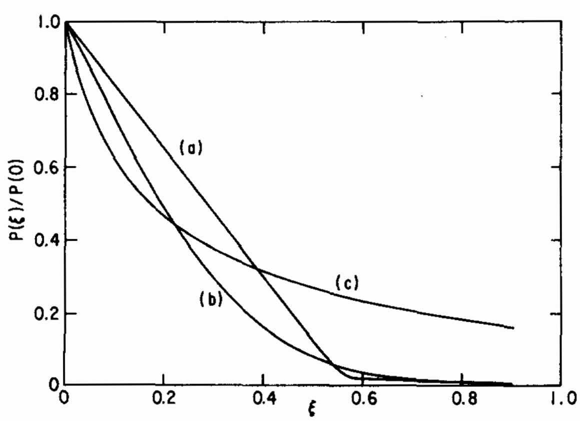

FIG. 2. Fraction of incident power retained by the rf as a function of depth into the plasma; (a) resonance broadening omitted, (b) resonance broadening included and (c) resonance broadening partially taken into account.

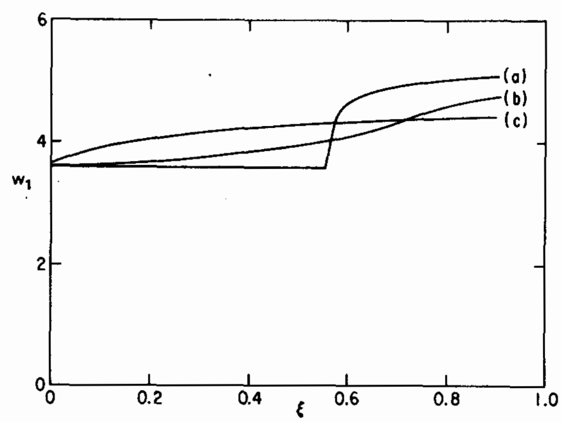

FIG. 4. Location of the inner (low velocity) edge of the rf spectrum; (a) resonance broadening omitted, (b) resonance broadening included, and (c) resonance broadening partially taken into account.

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APPENDIX

It can be shown that when the plasma is subject to the incident rf-electrostatic waves, the one-particle electron velocity distribution function satisfies the equation[7]

$$\frac{\partial}{\partial t} f(v_\perp^2, v_\parallel, t) = \frac{8\pi e^2}{m} \int d\mathbf{k} \; \mathcal{E}_\mathbf{k} \frac{k_\parallel^2}{k^2} \frac{\partial}{\partial v_\parallel} \times \frac{1}{-i\omega + ik_\parallel v_\parallel} \frac{\partial}{\partial v_\parallel} f(v_\perp^2, v_\parallel, t) + \left(\frac{\partial f}{\partial t}\right)_c , \tag{A1}$$

where $\perp$ and $\parallel$ refer to the direction of the magnetic field; v is the electron velocity, k is the wavenumber, ω is the wave frequency, and e and m are the charge and mass of the electron. The values of $\mathcal{E}_\mathbf{k}$, the spectral energy density, and $\mathcal{E}_{k_\parallel}$, the parallel spectral energy density, are expressed in terms of the energy density $\mathcal{W}$ associated with the incident rf waves (with electric field amplitude E) by the relationship

$$\int d\mathbf{k} \; \mathcal{E}_\mathbf{k} = \int dk_\parallel \; \mathcal{E}_{k_\parallel} = \mathcal{W} = \overline{E^2}/8\pi . \tag{A2}$$

The dispersion relation for lower hybrid waves, i.e.,

$$\omega^2 = \omega_{lh}^2 \left(1 + \frac{k_\parallel^2}{k^2}\frac{M}{m}\right), \quad \omega_{lh}^2 \approx \frac{\omega_{pi}^2}{1 + \omega_{pe}^2/\omega_{ce}^2} , \tag{A3}$$

where M is the ion mass and where ω_{pi}, ω_{pe}, and ω_{ce} are the plasma ion, plasma electron, and electron cyclotron frequencies, respectively, is used to eliminate $k_\parallel/k$ from the integrand in Eq. (A1). Therefore, the equation for the distribution function becomes

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial w} D(w) \frac{\partial}{\partial w} f + \left(\frac{\partial f}{\partial \tau}\right)_c , \tag{A4}$$

where we have expressed time in units of ν^{-1} defined by $\nu^{-1} = 4\pi n v_{te}^3 / 3\omega_{pe}^4 \ln\lambda$ and where

$$w = v_\parallel / v_{te} , \quad v_{te}^2 = \kappa T_e/m ,$$

$$D = \frac{D_0}{w} \mathcal{E}_{k_\parallel}\Big|_{\omega/wv_{te}} , \quad D_0 = \frac{2\pi \omega_{pi}^2 (\omega^2 - \omega_{lh}^2)}{nm\nu v_{te}^3 \omega_{lh}^2} . \tag{A5}$$

In computing the collision term, we assume that the background distributions of both the electrons and ions are nondrifting, nonevolving Maxwellian distributions.[8] Therefore, in the high velocity limit, valid for the resonant and nearby electrons, the collision term becomes

$$\frac{\partial f}{\partial \tau} = \frac{Z_i + 1}{6u^3} \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} + \frac{1}{3u^2}\frac{\partial}{\partial u}\left(\frac{1}{u}\frac{\partial f}{\partial u} + f\right) , \tag{A6}$$

where $u = v/v_{te}$ and $\mu = w/u$. It has been shown[4] that the perpendicular velocity space dynamics play a minor role. Thus, we assume a Maxwellian perpendicular distribution and integrate over that direction[9] so that Eq. (A4) becomes

$$\frac{\partial F(w)}{\partial \tau} = \frac{\partial}{\partial w} D(w) \frac{\partial}{\partial w} F(w) + \frac{2 + Z_i}{3} \frac{\partial}{\partial w}\left(\frac{1}{w^3}\frac{\partial}{\partial w} + \frac{1}{w^2}\right) F(w) , \tag{A7}$$

where $F(w) = \int f \, d\mathbf{v}_\perp$. When $Z_i = 1$, Eq. (A7) reduces to Eq. (1) with $(\partial F/\partial \tau)_c$ given by Eq. (3).

The decay of $D(w)$ is governed by the equation

$$\frac{\partial D}{\partial t} + v_{gx} \frac{\partial D}{\partial x} = 2\gamma' D , \tag{A8}$$

where the decay constant γ' and the group velocity v_{gx} for the lower hybrid electrostatic waves are

$$\gamma' = \frac{\pi}{2}\frac{\omega_{pe}^2}{\omega} w|w| \frac{k_\parallel^2}{k_\perp^2}\frac{\partial F}{\partial w} = \frac{\pi}{2}\frac{\omega_{pe}^2}{\omega} w|w| \left(\frac{\omega^2}{\omega_{lh}^2} - 1\right)\frac{m}{M}\frac{\partial F}{\partial w} ,$$

$$v_{gx} = Uw , \quad U = \frac{\omega_{lh}^2}{\omega^2}\left(\frac{\omega^2}{\omega_{lh}^2} - 1\right)^{3/2}\left(\frac{m}{M}\right)^{1/2} v_{te} . \tag{A9}$$

We express t in units of ν^{-1}, v_{gx} in units of U defined in Eq. (A9), and x in units of U/ν. Therefore, Eq. (A8) becomes

$$\frac{\partial D}{\partial \tau} + V_g \frac{\partial D}{\partial \xi} = \gamma D , \tag{A10}$$

where

$$V_g = w , \quad \xi = x\nu/U , \quad \gamma = 2\nu^{-1}\gamma' . \tag{A11}$$

From Eq. (A9) it follows that the wave damping rate can be written

$$\gamma = Aw|w|\frac{\partial F}{\partial w} \quad A = \nu^{-1}\pi\frac{\omega_{pi}^2}{\omega}\left(\frac{\omega^2}{\omega_{lh}^2} - 1\right) . \tag{A12}$$

When resonance broadening on F is included, Eq. (A12) reduces to Eq. (4).

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