Velocity-Space Diffusion in a Perpendicularly Propagating Electrostatic Wave

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Abstract

The motion of ions in the fields $\mathbf{B} = B_0\hat{\mathbf{z}}$ and $\mathbf{E} = E_0\hat{\mathbf{y}}\cos(k_\perp y - \omega t)$ is considered. When $\omega \gg \Omega_i$ and $v_\perp \gtrsim \omega/k_\perp$, the equations of motion may be reduced to a set of difference equations. These equations exhibit stochastic behavior when E_0 exceeds a threshold. The diffusion coefficient above the threshold is determined. Far above the threshold, ion Landau damping is recovered. Extension of the method to include parallel propagation is outlined.

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Equations of Motion

Consider an ion in a uniform magnetic field and a perpendicularly propagating electrostatic wave,

$$\mathbf{B} = B_0\hat{\mathbf{z}}, \qquad \mathbf{E} = E_0\hat{\mathbf{y}}\cos(k_\perp y - \omega t). \tag{1}$$

Normalizing lengths to $k_\perp^{-1}$ and times to Ω_i^{-1} ($\Omega_i = q_i B_0/m_i$), the Lorentz force law for the ion becomes

$$\ddot{y} + y = \alpha\cos(y - \nu t), \qquad \dot{x} = y, \tag{2}$$

where

$$\nu = \omega/\Omega_i, \tag{3a}$$

$$\alpha = \frac{E_0/B_0}{\Omega_i/k_\perp}. \tag{3b}$$

We solve (2) by approximating the force due to the wave by impulses at those points where the phase is slowly varying, i.e., at $\dot{y} = \nu$. The trajectory of the ion is given in Fig. 1. Expanding the trajectory about the resonance point, we find that the magnitude of the impulses is given by

$$\begin{aligned} B &= \int_{-\infty}^{\infty} \alpha\cos(\phi_c - \tfrac{1}{2}y_c t'^2)dt' \\ &= \alpha(2\pi/|y_c|)^{1/2}\cos[\phi_c - \mathrm{sign}(y_c)\pi/4], \end{aligned} \tag{4}$$

where $\phi_c = y_c - \nu t_c$ and t_c and y_c are the time and position of the wave-particle "collision." We may determine the Larmor radius and phase of the ion at the end of the jth orbit [the beginning of the $(j+1)$th orbit] in terms of these quantities at the beginning of the jth orbit. (Details are given in Ref. 1.) The resulting difference equations are

$$u = \theta - \rho, \qquad v = \theta + \rho, \tag{5a}$$

$$\theta = \tfrac{1}{2}(v + u), \qquad \rho = \tfrac{1}{2}(v - u), \tag{5b}$$

$$u_{j+1} - u_j = 2\pi\delta - 2\pi A\cos v_j, \tag{5c}$$

$$v_{j+1} - v_j = 2\pi\delta + 2\pi A\cos u_{j+1}, \tag{5d}$$

where

$$\theta = \nu t \pmod{2\pi}, \tag{6a}$$

$$\rho = (r^2 - \nu^2)^{1/2} - \nu\cos^{-1}(\nu/r) + \nu\pi - \pi/4, \tag{6b}$$

$$A = \left(\frac{2}{\pi}\right)^{1/2}\frac{\alpha\nu(r^2 - \nu^2)^{1/4}}{r^2}, \tag{7a}$$

$$\delta = \nu - n. \tag{7b}$$

Here r is the normalized Larmor radius, $k_\perp v_\perp/\Omega_i$, and n is an integer. The limits of validity of (5) are

$$\nu \gg 1, \qquad r-\nu \gg (\tfrac{1}{2}\nu)^{1/3}, \qquad A \ll (r^2-\nu^2)^{3/2}/r^2. \tag{8}$$

In Fig. 2 we compare the trajectories obtained using the exact equations of motion, (2), with those obtained from the difference equations, (5). We see that the agreement is very good indicating that (5) is an excellent approximation of (2).

There are three advantages to using the difference equations in preference to the Lorentz force law. Firstly, they are much quicker to solve numerically. Secondly, because of the way the equations were derived, the results are easier to interpret. Lastly, the equations have separated out two velocity-space scales, the ρ scale ($\sim \Omega_i/k_\perp$) and the r scale ($\sim \omega/k_\perp$). We therefore treat A which is a function of r as a constant when iterating the equations. This means that the diffusion coefficient is independent of ρ and so is much easier to determine numerically.

Examples of Trajectories

When A is infinitesimal, (5) may be solved by integrating (summing?) over unperturbed orbits. Substituting $u_j = u_0 + 2\pi\delta j$ and $v_j = v_0 + 2\pi\delta j$ into the right hand sides of (5c) and (5d) gives

$$\rho_N - \rho_0 = 2\pi A\cos(\rho_0 - \pi\delta) \times \begin{cases} (-1)^\delta \cos\theta_0\, N, & \text{for } \delta = \text{integer}, \\ \dfrac{\sin(2\pi\delta N + \theta_0) - \sin\theta_0}{2\sin(\pi\delta)}, & \text{for } \delta \neq \text{integer}. \end{cases} \tag{9}$$

Note that the trajectory is secular or not depending on whether or not δ is an integer (ω is a cyclotron harmonic). Formally, we may compute a diffusion coefficient using

$$\mathcal{D} = \operatorname*{Lim}_{N\to\infty} \frac{\langle(\rho_N - \rho_0)^2\rangle}{2N}. \tag{10}$$

Substituting (9) into (10) gives

$$\mathcal{D} = \pi^2 A^2 \cos^2\rho_0 \sum_{m=-\infty}^{\infty} \hat{\delta}(\delta - m), \tag{11}$$

where $\hat{\delta}$ is the Dirac delta function. Converting back to r and t and undoing the normalizations gives

$$D = \frac{\pi}{2}\frac{q_i^2 E_0^2}{m_i^2}\left(\frac{\omega}{k_\perp v_\perp}\right)^2 \sum_{m=-\infty}^{\infty} J_m^2\left(\frac{k_\perp v_\perp}{\Omega_i}\right)\hat{\delta}(\omega - m\Omega_i), \tag{12}$$

which is the usual quasi-linear diffusion coefficient.

If we consider finite but small A, then all the trajectories are bounded. There are three distinct cases, $\delta = 0$ (which is the case considered by Fukuyama *et al.*[2]), $\delta = \frac{1}{2}$, and $\delta \neq 0$ or $\frac{1}{2}$. The trajectories for these cases are shown in Fig. 3.

When A is increased, the system undergoes a stochastic transition, an example of which is shown in Fig. 4 for $\delta = 0.23$. Below the stochasticity threshold, nearly all the trajectories are integrable [Fig. 4(a)] or, if there are stochastic trajectories, they are bounded in ρ [Fig. 4(b)]. Above the threshold, nearly all the trajectories are stochastic and unbounded. The value of the threshold may be numerically determined and is found to be $A = A_s = \frac{1}{4}$. Above this value of A, the kick received by the ion during one transit through resonance is sufficient to change the phase of the kick received when next in resonance by $\pi/2$.

Diffusion Coefficient

When computing the diffusion coefficient numerically, it is convenient to work with the correlation function, C_k, where

$$C_k = \langle a_j a_{j+k} \rangle, \tag{13}$$

a_j is the particle acceleration, $a_j = \rho_{j+1} - \rho_j$, and the average is over an ensemble of particles and over the length of a given trajectory (i.e., over j). Then the diffusion coefficient, $\mathcal{D}$, is given by

$$\mathcal{D} = \frac{1}{2} C_0 + \sum_{k=1}^{\infty} C_k. \tag{14}$$

[This definition is equivalent to (10).] The advantages of defining $\mathcal{D}$ in this way are twofold. Firstly, the statistical fluctuations in the computation are minimized. Secondly, it is easy to introduce the effects of collisions on the diffusion coefficient. This is accomplished as follows: If k_0 is the mean number of cyclotron periods between decorrelating collisions (such collisions need only result in deflection by $1/\nu$, a small angle), then the probability of such a collision taking place in k periods is $1 - \exp(-k/k_0)$ since collisions are independent events. Collisions may then be included in the computations of $\mathcal{D}$ by replacing C_k in (14) by $C_k \exp(-k/k_0)$.

In the limit $A \gg A_s$, the kicks the ion receives are uncorrelated so that only C_0 is nonzero. Assuming that the trajectory is ergodic, we obtain $\mathcal{D} = \frac{1}{2}\pi^2 A^2$. When A is not large, we account for the correlations between the kicks received by the ion by writing

$$\mathcal{D} = \tfrac{1}{2}\pi^2 A^2 g^2(A), \tag{15}$$

Numerically determining $g(A)$ we find that approximately

$$g(A) = \max(1 - A_s^2/A^2, 0), \tag{16}$$

with $A_s = \frac{1}{4}$.

Converting (15) back into usual variables we obtain

$$D = \frac{1}{2} \frac{q_i^2 E_0^2}{m_i^2} \left(\frac{\omega}{k_\perp v_\perp} \right)^2 \frac{g^2(A)}{(k_\perp^2 v_\perp^2 - \omega^2)^{1/2}}. \tag{17}$$

In the limit $A \gg A_s$, when $g(A) \approx 1$, this is just the zero-magnetic-field result, $(\pi/2)(q_i/m_i)^2 E_0^2\, \hat{\delta}(\omega - k_\perp v_y)$, averaged over Larmor phase. Thus, in this limit, we recover ion Landau damping.

Extension to Parallel Propagation

The diffusion coefficient was so easily calculated above because of the simplification obtained by reducing the problem to difference equations, (5). This reduction may be achieved in similar problems. We consider here the case where the wave has some component of parallel propagation so that (1) becomes

$$\mathbf{B} = B_0\hat{\mathbf{z}}, \qquad \mathbf{E} = E_0(\hat{\mathbf{y}} + k_\parallel\hat{\mathbf{z}}/k_\perp)\cos(k_\perp y + k_\parallel z - \omega t). \tag{18}$$

Adopting the same normalization as before, we obtain

$$\ddot{y} + y = \alpha\cos(y + \zeta z - \nu t), \tag{19a}$$

$$\ddot{z} = \alpha\zeta\cos(y + \zeta z - \nu t), \tag{19b}$$

where $\zeta = k_\parallel/k_\perp$. The difference equations for a particle with normalized Larmor radius, r, and normalized parallel velocity, $w = k_\perp v_\parallel/\Omega_i$, are

$$u = \theta - \rho, \qquad v = \theta + \rho, \tag{20a}$$

$$u_{j+1} - u_j = 2\pi\gamma_{j+1/2} - 2\pi A\cos v_j, \tag{20b}$$

$$v_{j+1} - v_j = 2\pi\gamma_{j+1/2} + 2\pi A\cos u_{j+1}, \tag{20c}$$

$$\gamma_{j+1/2} = \delta - \beta(\rho_j + \pi A\cos v_j), \tag{20d}$$

where the variables θ and ρ are given by

$$\theta = \nu t - \zeta z \pmod{2\pi}, \tag{21a}$$

$$\rho = (r^2 - \mu^2)^{1/2} - \mu\cos^{-1}(\mu/r) + \mu\pi - \pi/4 - 2\pi m. \tag{21b}$$

The definitions of the parameters A, β, and ρ are

$$A = \left(\frac{2}{\pi}\right)^{1/2}\frac{\alpha\mu Q}{r(r^2 - \mu^2)^{1/4}}, \tag{22a}$$

$$\beta = \zeta^2 r/(\mu Q), \tag{22b}$$

$$\delta = \mu + \beta\rho - n. \tag{22c}$$

Here, m and n are integers and

$$\mu = \nu - \zeta w, \tag{23a}$$

$$Q = \frac{(r^2 - \mu^2)^{1/2}}{r} - \left[\pi - \cos^{-1}\left(\frac{\mu}{r}\right)\right]\frac{\zeta^2 r}{\mu}. \tag{23b}$$

Despite appearances δ is a parameter independent of ρ since the quantity $\mu + \beta\rho$ is a constant. (This follows from energy conservation in the wave frame.) The restrictions on the validity of (20) are

$$\mu \gg 1, \qquad r - \mu \gg (\tfrac{1}{2}\mu)^{1/3}. \tag{24}$$

A comparison between the exact equations of motion, (19), and the difference equations, (20), is shown in Fig. 5 for $\zeta = 1$ (propagation at 45°). Again, there is excellent agreement.

The results of Smith and Kaufman[3] may be obtained in the limit $\beta \to \infty$ and $A \to 0$ ($Q \to 0$). In that case, the change in ρ is negligible so that it is necessary to rescale the velocity variable by defining

$$\sigma_j = 4\pi\beta\rho_j - 2\pi\delta. \tag{25}$$

Equation (20) then becomes

$$\theta_{j+1} - \theta_j = -\sigma_j - \tfrac{1}{2}K\cos(\theta_j + \rho), \tag{26a}$$

$$\sigma_{j+1} - \sigma_j = \tfrac{1}{2}K[\cos(\theta_{j+1} - \rho) + \cos(\theta_j + \rho)], \tag{26b}$$

where $K = 4\pi^2 A\beta$. In (26) ρ is a constant. Setting

$$\psi = \sigma + \tfrac{1}{2}K cos(\theta + \rho) \tag{27}$$

gives

$$\theta_{j+1} - \theta_j = -\psi_j, \tag{28a}$$

$$\psi_{j+1} - \psi_j = K\cos\rho\cos\theta_{j+1}. \tag{28b}$$

This is the "standard mapping" studied by Chirikov.[4] The island overlap condition for this mapping is $|K\cos\rho| > \pi^2/4$ or

$$|\alpha| > \left|16\zeta^2 J_\mu(r)\right|^{-1}, \tag{29}$$

which is the stochasticity threshold obtained by Smith and Kaufman. The island overlap criterion is a significant overestimate of the stochasticity threshold for the standard mapping.[4] Greene[5] has calculated that the true threshold is a factor $(\pi^2/4)/0.971635 \approx 2\frac{1}{2}$ smaller than the result given above.

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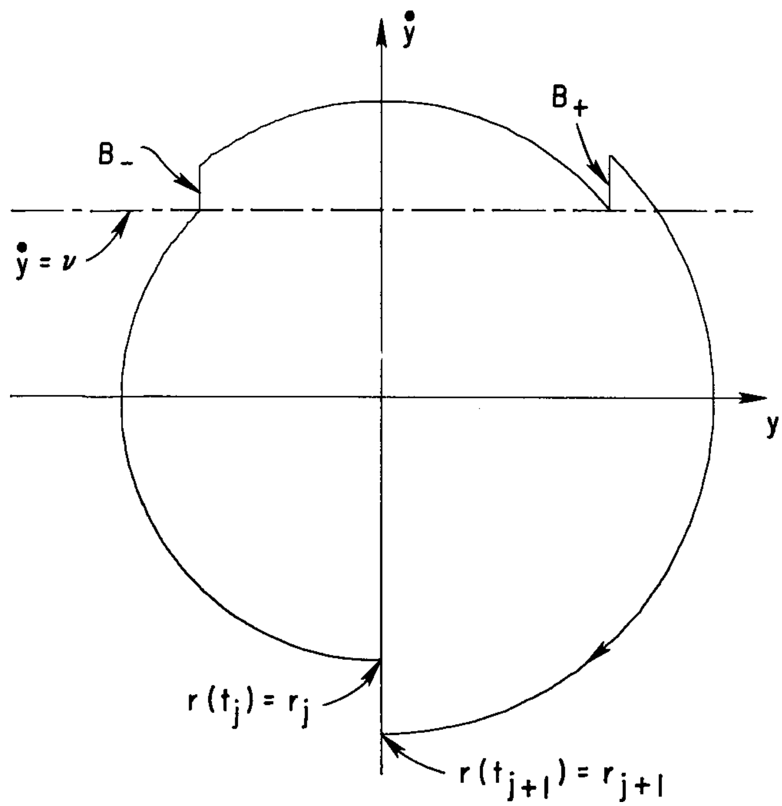

Fig. 1. Motion of an ion in velocity space, showing the kicks it receives when passing through wave-particle resonance.

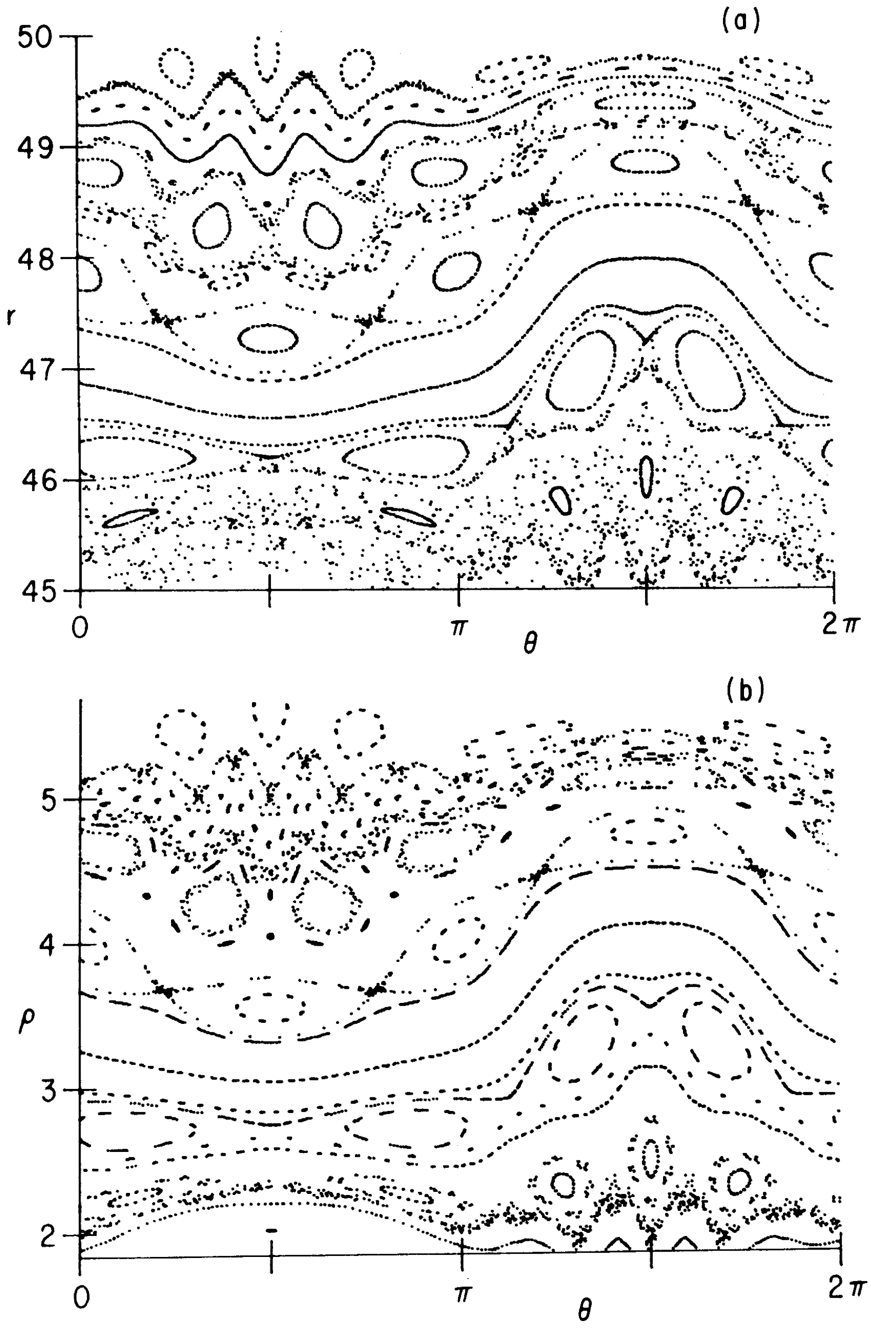

Fig. 2. Comparison of the difference equations with the Lorentz force law. (a) Trajectories computed using (2) with $\nu = 30.23$ and $\alpha = 2.2$. (b) Trajectories computed using (5) with $\delta = 0.23$ and $A = 0.1424$, which are given by (7) with $\nu = 30.23$, $\alpha = 2.2$, and $r = 47.5$. In each case the trajectories of 24 particles are followed for 300 orbits.

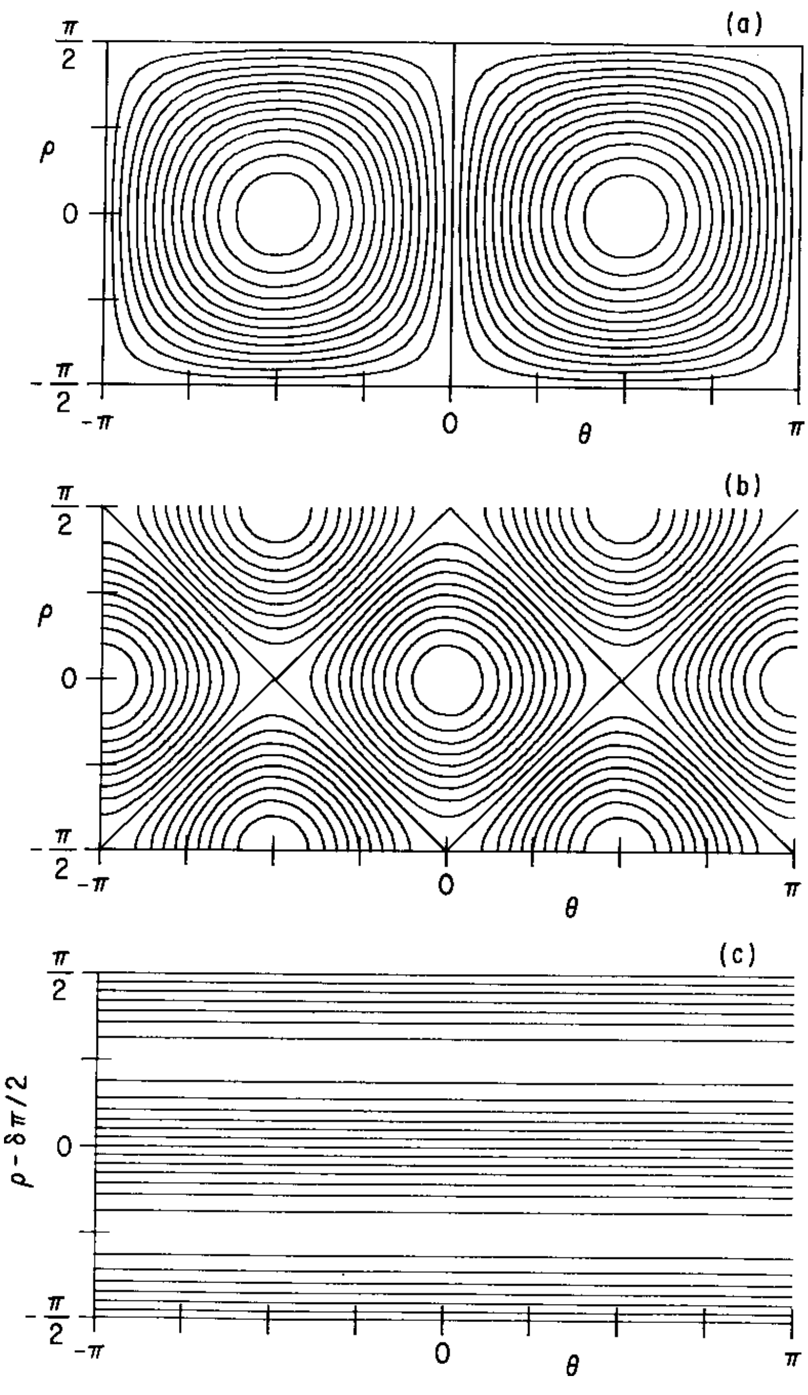

Fig. 3. Trajectories for small A and (a) $\delta = 0$, (b) $\delta = \frac{1}{2}$, and (c) $\delta \neq 0$ or $\frac{1}{2}$.

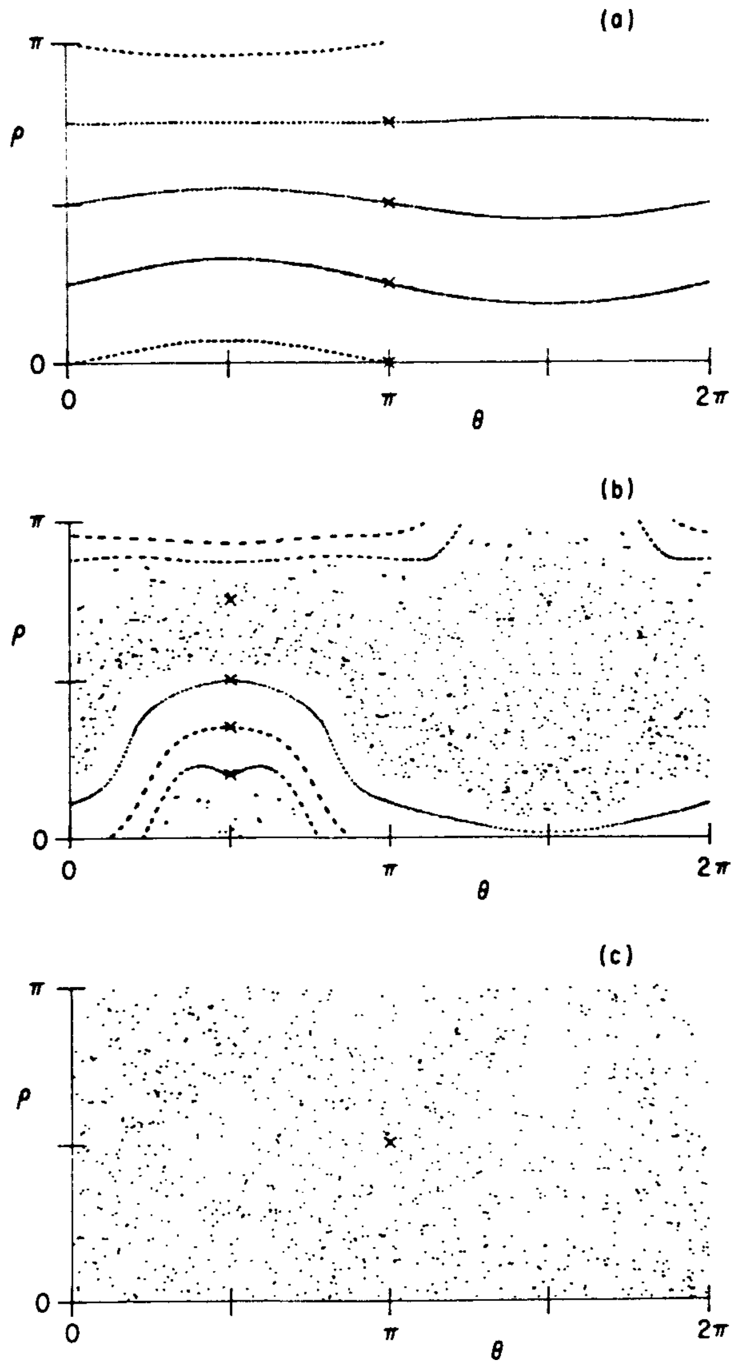

Fig. 4. The trajectories of ions with $\delta = 0.23$ and (a) $A = 0.05$, (b) $A = 0.2$, (c) $A = 0.35$. The initial positions are shown by crosses.

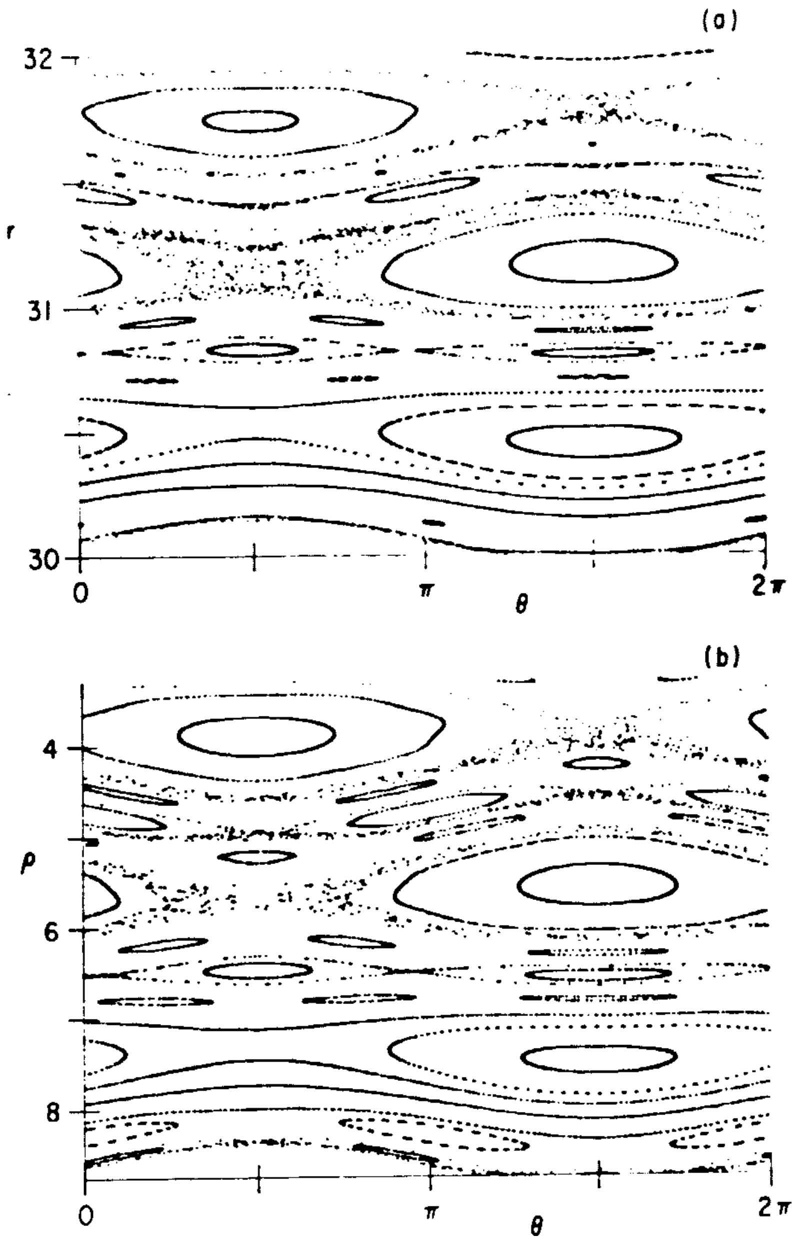

Fig. 5. Comparison of the difference equations with the Lorentz force law for finite $k_{\|}$. (a) Trajectories computed using (19) with $\nu = 20.23$, $\alpha = 0.23$, and $\zeta = 1$. The total energy (kinetic + electrostatic) of each particle is the same and is chosen so that $w \approx 0$ when $r = 31$. (b) Trajectories computed using (20) with $\delta = 0.8548$, $A = -0.06768$, and $\beta = -0.5595$ which are given by (22) with $\nu = 20.23$, $\alpha = 0.23$, $\zeta = 1$, $r = 31$, $w = 0$, and $m = 10$. In each case the trajectories of 24 particles are followed for 300 orbits. The vertical scale is inverted in (b) since $Q < 0$.