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DIFFERENTIAL FORM OF THE COLLISION INTEGRAL FOR A RELATIVISTIC PLASMA

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The differential formulation of the Landau-Fokker-Planck collision integral is developed for the case of relativistic electromagnetic interactions.

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Kinetic theory is founded upon the Boltzmann equation, which is a conservation equation for the phase-space distribution function of each species in an ensemble of interacting particles. For the case of Coulomb interactions, Landau[1] expressed the collision term in the Fokker-Planck form. This mixed integro-differential representation was extended to relativistic electromagnetic interactions by Beliaev and Budker.[2] For the nonrelativistic case, it was shown by Rosenbluth et al.[3] and by Trubnikov[4] that the integrals appearing in the collision term can be expressed in terms of the solution of a pair of differential equations. The present work extends that formulation to the relativistic collision integral. Using an expansion in spherical harmonics the relativistic differential formulation is then applied to calculate the scattering and slowing down of fast particles in a relativistic equilibrium background plasma. Our work is relevant to the study of high temperature plasma in fusion energy research and in astrophysics.

In the work of Landau[1] and that of Beliaev and Budker,[2,5] the collision term that occurs on the right-hand side of the Boltzmann equation for species a and describes the effect of collisions with species b is written in the Fokker-Planck form,

$$C_{ab} = \frac{\partial}{\partial \mathbf{u}} \cdot \left(\boldsymbol{D}_{ab} \cdot \frac{\partial f_a}{\partial \mathbf{u}} - \mathbf{F}_{ab} f_a\right), \tag{1}$$

in which the coefficients $\boldsymbol{D}_{ab}$ and $\mathbf{F}_{ab}$ are defined by

$$\boldsymbol{D}_{ab}(\mathbf{u}) = \frac{q_a^2 q_b^2}{8\pi\epsilon_0^2 m_a^2} \log\Lambda_{ab} \int \boldsymbol{U}(\mathbf{u},\mathbf{u}') f_b(\mathbf{u}')\, d^3\mathbf{u}', \tag{2a}$$

$$\mathbf{F}_{ab}(\mathbf{u}) = -\frac{q_a^2 q_b^2}{8\pi\epsilon_0^2 m_a m_b} \log\Lambda_{ab} \int \left(\frac{\partial}{\partial \mathbf{u}'} \cdot \boldsymbol{U}(\mathbf{u},\mathbf{u}')\right) f_b(\mathbf{u}')\, d^3\mathbf{u}'. \tag{2b}$$

Here, f_a and f_b are the distribution functions for the two species, $\mathbf{u}$ is the ratio of momentum to species mass, q_a and q_b are the species charge, m_a and m_b are the species mass, ϵ_0 is the vacuum dielectric permittivity, and $\log\Lambda_{ab}$ is the Coulomb logarithm. The kernel $\boldsymbol{U}$ is specified below. This form of the collision operator is only approximate because of the introduction of cutoffs in the collision integral. More accurate operators that take into account Debye shielding at large impact parameters and large-angle scattering and quantum effects at small impact parameters have been derived.[6,7] The purpose of this letter is to present a differential formulation for the integral transforms that occur in Eqs. (2). To avoid unnecessary clutter we discard the factor that depends only on the species properties, drop the species subscript, and consider the transforms

$$\boldsymbol{D}(\mathbf{u}) = \frac{1}{8\pi} \int \boldsymbol{U}(\mathbf{u},\mathbf{u}') f(\mathbf{u}')\, d^3\mathbf{u}', \tag{3a}$$

$$\mathbf{F}(\mathbf{u}) = -\frac{1}{8\pi} \int \left(\frac{\partial}{\partial \mathbf{u}'} \cdot \boldsymbol{U}(\mathbf{u},\mathbf{u}')\right) f(\mathbf{u}')\, d^3\mathbf{u}'. \tag{3b}$$

For guidance, let us recall briefly the nonrelativistic theory.[3,4] In that case the momentum-to-mass ratios $\mathbf{u}$ and $\mathbf{u}'$ reduce to the velocities $\mathbf{v}$ and $\mathbf{v}'$, and the collision kernel is the one given by Landau,[1] $\boldsymbol{U} = (|\mathbf{s}|^2\boldsymbol{I} - \mathbf{ss})/|\mathbf{s}|^3$, where $\mathbf{s} = \mathbf{v} - \mathbf{v}'$. It may be seen that $\boldsymbol{U} = \partial^2|\mathbf{s}|/\partial\mathbf{v}\partial\mathbf{v}$ and $(\partial/\partial\mathbf{v}')\cdot\boldsymbol{U} = -2\partial|\mathbf{s}|^{-1}/\partial\mathbf{v}$. To obtain the differential formulation, these representations are inserted into Eqs. (3), and the differentiation with respect to $\mathbf{v}$ is moved outside the integration over $\mathbf{v}'$. Defining the potentials $h(\mathbf{v}) = -(1/8\pi)\int|\mathbf{s}|f\,d^3\mathbf{v}'$ and $g(\mathbf{v}) = -(1/4\pi)\int|\mathbf{s}|^{-1}f\,d^3\mathbf{v}'$, we have $\boldsymbol{D} = -\partial^2h/\partial\mathbf{v}\partial\mathbf{v}$ and $\mathbf{F} = -\partial g/\partial\mathbf{v}$. Furthermore, from $\Delta|\mathbf{s}| = 2|\mathbf{s}|^{-1}$ and $\Delta|\mathbf{s}|^{-1} = -4\pi\delta(\mathbf{s})$ it follows that h and g obey the equations $\Delta h = g$ and $\Delta g = f$. (Δ denotes the Laplacian with respect to the variable $\mathbf{v}$.) These equations provide the differential formulation of the collision term in the nonrelativistic case.

The Landau collision kernel was obtained in a semi-relativistic fashion, assuming Coulomb collisions and relativistic particle kinematics. It is a good approximation to the fully relativistic kernel given below provided that $|\mathbf{v}.\mathbf{v}'| \ll c^2$, which is true when one of the colliding particles is nonrelativistic. However, the reduction of the collision integral to the differential form of Rosenbluth and Trubnikov relies on the stronger assumptions $|\mathbf{v}|^2 \ll c^2$ and $|\mathbf{v}'|^2 \ll c^2$, and is therefore entirely nonrelativistic. A differential formulation that is exactly equivalent to the Landau collision integral was given by Franz.[8]

We turn now to the differential formulation of the relativistic collision integral due to Beliaev and Budker.[2,5,6] They obtained the expression

$$\boldsymbol{U}(\mathbf{u},\mathbf{u}') = \frac{r^2/(\gamma\gamma')}{(r^2-1)^{3/2}}\big((r^2-1)\boldsymbol{I} - \mathbf{uu} - \mathbf{u}'\mathbf{u}' + r(\mathbf{uu}' + \mathbf{u}'\mathbf{u})\big), \tag{4a}$$

in which $\gamma = \sqrt{1+|\mathbf{u}|^2}$, $\gamma' = \sqrt{1+|\mathbf{u}'|^2}$, and $r = \gamma\gamma' - \mathbf{u}.\mathbf{u}'$. (We set $c = 1$ in this part of the paper.) One finds

$$\frac{\partial}{\partial\mathbf{u}'}\cdot\boldsymbol{U}(\mathbf{u},\mathbf{u}') = \frac{2r^2/(\gamma\gamma')}{(r^2-1)^{3/2}}(r\mathbf{u} - \mathbf{u}'). \tag{4b}$$

Notice that r is the relativistic correction factor for the relative velocity between the two particles (i.e., for the velocity of one particle in the rest frame of the other). Conversely, this relative velocity is given by $r^{-1}(r^2-1)^{1/2}$.

In developing a differential formulation for the collision term based on the Beliaev and Budker kernel, it is helpful to work in terms of relativistically covariant quantities. The expression $\gamma\gamma'\boldsymbol{U}$ is equal to the space part of a four-tensor W that depends on the four-vectors $u = (\gamma, \mathbf{u})$ and $u' = (\gamma', \mathbf{u}')$,

$$W^{ij}(u,u') = \frac{r^2}{(r^2-1)^{3/2}}\big((r^2-1)g^{ij} - u^iu^j - u'^iu'^j + r(u^iu'^j + u'^iu^j)\big), \tag{5a}$$

where g^{ij} is the metric tensor, with signature $-{+}{+}{+}$. ($r = -u_iu'^i$ is clearly a four-scalar.) The tensor W is symmetric ($W^{ij} = W^{ji}$), symmetric in u and u', satisfies $u_iW^{ij} = 0$, and satisfies $W^i_i = 2r^2(r^2-1)^{-1/2}$. Likewise $\gamma\gamma'(\partial/\partial\mathbf{u}')\cdot\boldsymbol{U}$ is the space part of the four-vector V, where

$$V^i(u,u') = \frac{2r^2}{(r^2-1)^{3/2}}(ru^i - u'^i). \tag{5b}$$

If the relativistic differential formulation is to parallel most closely the nonrelativistic formulation, then one should find a representation of the form $W^{ij} = \mathcal{H}^{ij}\psi$ and $V^i = -2\mathcal{G}^i\varphi$, where ψ and φ are four-scalars depending on u and u', and $\mathcal{H}^{ij}$ and $\mathcal{G}^i$ are covariant differential operators acting on the variable u. In the nonrelativistic limit, ψ should reduce to $|\mathbf{v} - \mathbf{v}'|$ and φ should reduce to $|\mathbf{v} - \mathbf{v}'|^{-1}$. It should be possible to transform ψ and φ to delta functions by a sequence of second-order differential operators. The potentials would be defined as $h = -(1/8\pi)\int(\psi f/\gamma')\,d^3\mathbf{u}'$ and $g = -(1/4\pi)\int(\varphi f/\gamma')\,d^3\mathbf{u}'$; these expressions define four-scalars (cf. Ref. 5). The differential equations satisfied by h and g follow immediately from those satisfied by ψ and φ. Finally, $\boldsymbol{D}$ would be obtained as the space part of $-\gamma^{-1}\mathcal{H}^{ij}h$ and $\mathbf{F}$ as the space part of $-\gamma^{-1}\mathcal{G}^i g$. In fact, it will turn out that the relativistic formulation has to be somewhat more complicated, but not fundamentally different from the outline just sketched.

A function of the four-vectors u and u' that is a four-scalar must be a function of $r = -u.u'$ alone. The form of the differential operators $\mathcal{H}^{ij}$ and $\mathcal{G}^i$ is restricted because these should be interior operators on the surface $u^2 = -1$ in four-space. In addition, it is required that $\mathcal{H}^{ij} = \mathcal{H}^{ji}$ and $u_i\mathcal{H}^{ij} = 0$. Under those restrictions it is found that the most general form of $\mathcal{H}^{ij}$ and $\mathcal{G}^i$, up to a multiplicative constant, is $\mathcal{H}^{ij}\chi = \mathcal{L}^{ij}\chi + \alpha(g^{ij} + u^i u^j)\chi$ and $\mathcal{G}^i\chi = \mathcal{K}^i\chi + \beta u^i\chi$. Here, α and β are arbitrary constants, and

$$\mathcal{L}^{ij}\chi = (g^{ik} + u^i u^k)(g^{jl} + u^j u^l)\frac{\partial^2\chi}{\partial u^k \partial u^l} + (g^{ij} + u^i u^j)u^m\,\frac{\partial\chi}{\partial u^m}\,, \tag{6a}$$

$$\mathcal{K}^i\chi = (g^{ik} + u^i u^k)\,\frac{\partial\chi}{\partial u^k}\,. \tag{6b}$$

The spatial part of $\mathcal{L}^{ij}\chi$ is $\boldsymbol{L}\chi$ and that of $\mathcal{K}^i\chi$ is $\mathbf{K}\chi$ where

$$\boldsymbol{L}\chi = \gamma^{-2}\frac{\partial^2\chi}{\partial\mathbf{v}\partial\mathbf{v}} - \mathbf{v}\,\frac{\partial\chi}{\partial\mathbf{v}} - \frac{\partial\chi}{\partial\mathbf{v}}\,\mathbf{v}, \tag{7a}$$

$$\mathbf{K}\chi = \gamma^{-1}\,\frac{\partial\chi}{\partial\mathbf{v}}\,, \tag{7b}$$

in which $\mathbf{v} = \mathbf{u}/\gamma$, and $\partial/\partial\mathbf{v} = \gamma(\boldsymbol{I} + \mathbf{uu})\cdot\partial/\partial\mathbf{u}$. If χ is a function of r alone then

$$\mathcal{L}^{ij}\chi = \frac{d^2\chi}{dr^2}(ru^i - u'^i)(ru^j - u'^j) + r\frac{d\chi}{dr}(g^{ij} + u^i u^j)$$

and $\mathcal{K}^i\chi = (d\chi/dr)(ru^i - u'^i)$. One is thereby led to the representations

$$\begin{aligned} W^{ij} &= \big[\mathcal{L}^{ij} + g^{ij} + u^i u^j\big]\sqrt{r^2-1} \\ &\quad - \big[\mathcal{L}^{ij} - g^{ij} - u^i u^j\big]\big(r\cosh^{-1}r - \sqrt{r^2-1}\big), \qquad (8a)\\ V^i &= -2\mathcal{K}^i\big(r(r^2-1)^{-1/2} - \cosh^{-1}r\big). \qquad (8b) \end{aligned}$$

These representations for W and V are only suitable for constructing a differential formulation of the collision term if the functions that occur on the right-hand sides can be reduced to delta functions by some sequence of differential operators. For that purpose the contraction $L = \mathcal{L}^i_i$ is needed; in terms of the three-space variables it is

$$L\chi = (\boldsymbol{I} + \mathbf{uu}) : \frac{\partial^2\chi}{\partial\mathbf{u}\partial\mathbf{u}} + 3\mathbf{u}\cdot\frac{\partial\chi}{\partial\mathbf{u}}\,. \tag{9}$$

If χ is a function of r alone, then $L\chi = (r^2-1)(d^2\chi/dr^2) + 3r(d\chi/dr)$ away from $r = 1$; at $r = 1$ (or $\mathbf{u} = \mathbf{u}'$) there may be a singularity. Specifically, it is found that

$$\begin{aligned} L\big(r(r^2-1)^{-1/2}\big) &= -4\pi\gamma\delta(\mathbf{u} - \mathbf{u}'),\\ [L+1]\big((r^2-1)^{-1/2}\big) &= -4\pi\gamma\delta(\mathbf{u} - \mathbf{u}'),\\ L\big(\cosh^{-1}r\big) &= 2r(r^2-1)^{-1/2},\\ [L-3]\big(\sqrt{r^2-1}\big) &= 2(r^2-1)^{-1/2},\\ [L-3]\big(r\cosh^{-1}r - \sqrt{r^2-1}\big) &= 4\sqrt{r^2-1}. \end{aligned}$$

The explicit form of the differential representation of Eqs. (3) based on the Beliaev and Budker collision kernel[2] follows: The potentials are

$$h_0 = -(1/4\pi)\int (r^2-1)^{-1/2} f(\mathbf{u}')/\gamma'\, d^3\mathbf{u}', \tag{10a}$$

$$h_1 = -(1/8\pi)\int \sqrt{r^2-1}\, f(\mathbf{u}')/\gamma'\, d^3\mathbf{u}', \tag{10b}$$

$$h_2 = -(1/32\pi)\int \bigl(r\cosh^{-1}r - \sqrt{r^2-1}\bigr) f(\mathbf{u}')/\gamma'\, d^3\mathbf{u}', \tag{10c}$$

$$g_0 = -(1/4\pi)\int r(r^2-1)^{-1/2} f(\mathbf{u}')/\gamma'\, d^3\mathbf{u}', \tag{10d}$$

$$g_1 = -(1/8\pi)\int \cosh^{-1}r\, f(\mathbf{u}')/\gamma'\, d^3\mathbf{u}'. \tag{10e}$$

These potentials satisfy the differential equations

$$\begin{aligned}[] [L+1]h_0 &= f, \\ [L-3]h_1 &= h_0, \\ [L-3]h_2 &= h_1, \end{aligned} \qquad \begin{aligned} Lg_0 &= f, \\ Lg_1 &= g_0. \end{aligned} \tag{11}$$

Finally one obtains $\boldsymbol{D}$ and $\mathbf{F}$ as

$$\boldsymbol{D}(\mathbf{u}) = -\gamma^{-1}[\boldsymbol{L} + \boldsymbol{I} + \mathbf{u}\mathbf{u}]h_1 + 4\gamma^{-1}[\boldsymbol{L} - \boldsymbol{I} - \mathbf{u}\mathbf{u}]h_2, \tag{12a}$$

$$\mathbf{F}(\mathbf{u}) = -\gamma^{-1}\mathbf{K}(g_0 - 2g_1). \tag{12b}$$

Equations (11–12) together with the definitions, Eqs. (7) and (9), provide the differential formulation in the relativistic case.

In order to proceed further analytically, it is useful to decompose the distribution function and the potentials in spherical harmonics, e.g.,

$$f(u,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}(u) P_n^m(\cos\theta)\exp(im\phi). \tag{13}$$

Here $u = |\mathbf{u}|$ (different from the convention used earlier), θ is the polar angle, and ϕ is the azimuthal angle. The equation $[L-\alpha]g = f$ is equivalent to the system of separated equations $[L_n - \alpha]g_{nm} = f_{nm}$, where

$$[L_n - \alpha]y = (1+u^2)\frac{d^2y}{du^2} + (2u^{-1} + 3u)\frac{dy}{du} - \left(\frac{n(n+1)}{u^2} + \alpha\right)y. \tag{14}$$

After the change of variable $x = \sinh^{-1}u$ and the change of unknown $z = (\sinh x)^{-n}y$, then the equation $[L_n - \alpha]y = w$ transforms to $[\mathcal{D}_n - a^2]z = (\sinh x)^{-n}w$, where $a^2 = \alpha + 1$ and

$$[\mathcal{D}_n - a^2]z = \frac{d^2z}{dx^2} + 2(n+1)(\coth x)\frac{dz}{dx} + \bigl((n+1)^2 - a^2\bigr)z. \tag{15}$$

The solution to the homogeneous equation $[\mathcal{D}_n - a^2]z = 0$ is required in order to construct a Green's function for the problem. To obtain this solution we note the following recurrence: If $z_{n-1,a}$ solves $[\mathcal{D}_{n-1} - a^2]z = 0$, then $z_{n,a} = (\sinh x)^{-1}(d/dx)z_{n-1,a}$ solves $[\mathcal{D}_n - a^2]z = 0$. Furthermore, for $n = -1$ the homogeneous equation is trivial to solve. However, the recurrence breaks down in the case that a is an integer. If $a = n$, then $z_{n-1,a} = 1$ solves

$[\mathcal{D}_{n-1}-a^2]z=0$, and differentiation produces the null solution to $[\mathcal{D}_n-a^2]z=0$. The recurrence must then be restarted from the general solution to $[\mathcal{D}_n-n^2]z=0$, which is

$$z_{n,n}=(\sinh x)^{-2n-1}\biggl(C_1+C_2\int_0^x(\sinh x')^{2n}dx'\biggr).$$

The integral that occurs here can be expressed in closed form.

The Green's function allows us to reduce the separated ordinary differential equations to quadrature. An important special application for these results is in the treatment of collisions off an equilibrium background distribution. Assuming that f_b is a stationary Maxwellian with density n_b and temperature T_b and that the energy of the colliding particles greatly exceeds T_b, we obtain

$$D_{uu}=\Gamma_{ab}\frac{K_1}{K_2}\frac{u_{tb}^2}{v^3}\biggl(1-\frac{K_0}{K_1}\frac{u_{tb}^2}{\gamma^2c^2}\biggr),$$
$$D_{\theta\theta}=\Gamma_{ab}\frac{1}{2v}\biggl[1-\frac{K_1}{K_2}\biggl(\frac{u_{tb}^2}{u^2}+\frac{u_{tb}^2}{\gamma^2c^2}\biggr)+\frac{K_0}{K_2}\frac{u_{tb}^2}{u^2}\frac{u_{tb}^2}{\gamma^2c^2}\biggr],$$

and $F_u=-(m_av/T_b)D_{uu}$. (The other components of $\boldsymbol{D}$ and $\mathbf{F}$ vanish.) Here we have put the expressions for $\boldsymbol{D}$ and $\mathbf{F}$ into dimensional form as in Eqs. (2), K_n is the nth-order Bessel function of the second kind, the argument for the Bessel functions is m_bc^2/T_b, $u_{tb}^2=T_b/m_b$, and $\Gamma_{ab}=n_bq_a^2q_b^2\log\Lambda_{ab}/(4\pi\epsilon_0^2m_a^2)$. The errors are exponentially small in u/u_{tb}.

To conclude, we have presented a differential formulation for the Beliaev and Budker[2] relativistic collision integral. This permits the rapid numerical evaluation of the collision term. A decomposition into spherical harmonics is useful in carrying out analytical work. It also provides a convenient method for calculating the boundary conditions for the potentials.

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