Alfvén Wave Heating with High-Q Eigenmodes*

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We consider the equation governing the mode structure of Alfvén waves. Since the Alfvén speed is a function of density, the compressional Alfvén wave changes from evanescent to propagating at some point in the plasma. However the presence of the shear Alfvén resonance changes this cutoff into a cutoff-resonance-cutoff triplet. By analytically and numerically studying the properties of the governing equation, we determine the absorption due to the resonance and Q for the eigenmode. We show that heating is possible using relatively high-Q eigenmodes.

The dispersion relation for Alfvén waves may be calculated using the cold two-fluid equations, assuming quasineutrality. The result is

$$D(\omega, \mathbf{k}) = (A - k^2)(A - k_{\parallel}^2) - D^2 \tag{1}$$

where $A = k_A^2 \Omega_i^2/(\Omega_i^2 - \omega^2)$, $D = (\omega/\Omega_i)A$, $k_A = \omega/v_A$, $v_A = c\Omega_i/\omega_{pi} = B/(\mu_0 n m_i)^{1/2}$. Consider a slab model with $\mathbf{B}$ constant and parallel to $\hat{\mathbf{z}}$ and the density gradient parallel to $\hat{\mathbf{x}}$. Then if we impose k_y and k_z the solution of (1) for k_x is

$$k_x^2 = \frac{(A + D - k_z^2)(A - D - k_z^2)}{A - k_z^2} - k_y^2. \tag{2}$$

It is clear from (2) that as the wave penetrates the plasma (note A, $D \propto n$) it passes a cutoff followed by a resonance and then a second cutoff. The wave is propagating on the high density side and evanescent on the low density side of this cutoff-resonance-cutoff triplet.

In order to determine the effect of the resonance on the propagation we convert (2) back into a differential equation,

$$(d^2/dx^2)\phi + k_x^2\phi = 0 \tag{3}$$

where k_x is given by (2) and ϕ is an appropriate mixture of the various field amplitudes. We will assume a linear density gradient, $n(x) = n_0 x/L$, where n_0 is the maximum density. Taking $k_A(x_R) = k_z$ and introducing a rescaled independent coordinate, $X = (x - x_R)k_{A0}^{2/3}/L^{1/3}$, where $k_{A0} = k_A(L)$, we obtain

$$\frac{d^2\phi}{dX^2} + \left[\frac{(X - N^2 s)(X + N^2 s)}{X + N^2 s^2} - M^2\right]\phi = 0 \tag{4}$$

where $N = k_z L^{1/3}/k_{A0}^{2/3}$, $M = k_y L^{1/3}/k_{A0}^{2/3}$ and $s = \omega/\Omega_i$. We consider the case $s < 1$. If the shear Alfvén resonance is not too close to the center of the plasma then $N < 1$. Therefore for simplicity we replace the denominator in (4) by X. We may write (4) in a generic form,

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$$\frac{d^2\phi}{dX^2} + \frac{(X + \delta_1)(X - \delta_2)}{X}\phi = 0 \tag{5}$$

where $\delta_2 = \frac{1}{2}M^2 + (\frac{1}{4}M^4 + N^4 s^2)^{1/2}$ and $\delta_1 = N^4 s^2/\delta_2$.

The solution to (5) is singular at $X = 0$. As $|X| \to \infty$, $\phi \to \mathrm{Ai}[-(X + \delta_1 - \delta_2)]$ or $\mathrm{Bi}[-(X + \delta_1 - \delta_2)]$. We will start at $X = -\infty$ with the Ai solution, which decays as $X \to -\infty$. Then by integrating above the singularity at $X = 0$ we obtain $\phi = A\,\mathrm{Ai}[-(X + \delta_1 - \delta_2)] + B\,\mathrm{Bi}[-(X + \delta_1 - \delta_2)]$ at $X = \infty$. (We determine that we must go above the singularity by letting ω have a small positive imaginary part, corresponding to a growing wave.) By regarding this mixture of solutions as the sum of an incident wave (from the right) and a reflected wave we may determine the fraction of the incident energy absorbed as

$$q = 4\,\mathrm{Im}(A^*B)/|A - iB|^2 \approx 4\,\mathrm{Im}(B/A) \tag{6}$$

for $|B| \ll |A|$. Note that $q > 0$ corresponds to dissipation.

In solving (5) we treat $N^2 s$ as small. We can then distinguish two cases of interest:

(1) $M \ll 1$, $\delta_2 \ll 1$, $\delta_1 \ll 1$

(2) $M \gg 1$, $\delta_2 = M^2 \gg 1$, $\delta_1\delta_2 = N^4 s^2 \ll 1$

Case 1. $\delta_1, \delta_2 \ll 1$

For $|X| \gg 1$ we have $\phi = \mathrm{Ai}(-X)$ or $\mathrm{Bi}(-X)$. We solve (5) in the connection region by developing series solutions about the singular point $X = 0$,

$$\phi = \phi_1(X) = f(-X) - \delta_1\delta_2 \log(X) g(-X) + O(\delta^2)$$

$$\phi = \phi_2(X) = -\,g(-X) + O(\delta^2)$$

where $c_1 f(-X) = \mathrm{Bi}(-X) + \sqrt{3}\,\mathrm{Ai}(-X)$, $c_2 g(-X) = \mathrm{Bi}(-X) - \sqrt{3}\,\mathrm{Ai}(-X)$, $c_1 = \mathrm{Ai}(0)$, $c_2 = -\,\mathrm{Ai}'(0)$. For large X we may treat $\log(X)$ as a constant, in which case ϕ_1 and ϕ_2 are linear combinations of the Airy functions. If we do the connection of the solutions at $X = \pm X_0$, we find that the solution which becomes $\mathrm{Ai}(-X)$ as $X \to -\infty$ is

$$\phi = \mathrm{Ai}(-X) + i\,1.24\,\delta_1\delta_2\,\mathrm{Bi}(-X) + O(\delta^2) \tag{7}$$

for $X \to \infty$, where $1.24 = \pi c_1/(2\sqrt{3}c_2)$. (Here we have only retained the $O(\delta^2)$ term which contributes to q.) Thus from (6) the dissipation is

$$q = 5\,\delta_1\delta_2 \tag{8}$$

Case 2. $\delta_2 \gg 1$, $\delta_1\delta_2 \ll 1$

In this case the probem simplifies since, far from the cutoff at $X = -\delta_1$ and the resonance at $X = 0$, the solution is $\mathrm{Ai}(\delta_2 - X)$ or $\mathrm{Bi}(\delta_2 - X)$, while close to these points the factor $X - \delta_2$ in (5) is approximately $-\delta_2$. In the latter case the resulting equation is one for which the solution may be written as an integral,

$$\phi = \int_C \frac{1}{t^2 - \kappa^2} \left[\frac{t + \kappa}{t - \kappa} \right]^{\epsilon} e^{Xt} \, dt \tag{9}$$

where $\epsilon = \frac{1}{2}\delta_1 \sqrt{\delta_2} \ll 1$ and $\kappa = \sqrt{\delta_2}$. This solution enables us to make the connection between the Airy solutions. The end points of C may be chosen where the integrand vanishes at $|t| \to \infty$. For $X < 0$ we pick $C = C_1$ shown in Fig. 1a, since the integral then is of the order of $e^{\kappa X}$ and so matches onto $\mathrm{Ai}(\delta_2 - X)$. For $X > 0$, C_1 becomes the contour shown in Fig. 1b. We split C_1 into two pieces C_a and C_b (see Fig. 1c). With appropriate choice of cuts the integral over C_a is purely imaginary, since the integrand takes on complex conjugate values either side of the real t axis. Letting $\epsilon \to 0$ in (9) we may evaluate the integral as the residue of the pole at $t = \kappa$ and obtain $\int_{C_a} \approx (i\pi/\kappa) e^{\kappa X}$. For the integral over C_b we let $(t - \kappa)^{\epsilon} = e^{-i\pi\epsilon} (\kappa - t)^{\epsilon}$. Taking the factor $e^{i\pi\epsilon}$ out of the integral the remaining terms again give a purely imaginary result for the integral. Thus $\int_{C_b} \approx (i\pi/\kappa) e^{i\pi\epsilon} e^{-\kappa X}$, where as before we have evaluated the integral by letting $\epsilon \to 0$. The solution for $X > 0$ is (dividing out $i\pi/\kappa$)

$$\phi = e^{\kappa X} + i\pi\epsilon e^{-\kappa X} .$$

(As before the terms we have neglected do not affect q.) Matching this onto the Airy functions gives

$$\phi = \mathrm{Ai}(\delta_2 - X)/\mathrm{Ai}(\delta_2) + i\pi\epsilon \, \mathrm{Bi}(\delta_2 - X)/\mathrm{Bi}(\delta_2) \tag{10}$$

so we obtain

$$q = 4\pi\epsilon \, \mathrm{Ai}(\delta_2)/\mathrm{Bi}(\delta_2) \approx \pi\delta_1 \sqrt{\delta_2} \exp(-\tfrac{4}{3}\delta_2^{3/2}) . \tag{11}$$

In Fig. 2 we compare the results (8) and (11) with the results obtained by numerically integrating (5). We note that if we use (8) for $\delta_2 < 0.3$ and (11) for $\delta_2 > 0.3$ we are in error by at most a factor of 2 in q.

Finally we note that q is the fractional energy lost per pass of the wave through the machine, whereas $1/Q$ is defined as the fractional energy lost per wave period. Thus Q is approximately ℓ/q where ℓ is the mode number in the x (or radial) direction. As an example consider a plasma with $B = 35\,\mathrm{kG}$, $n_0 = 3 \times 10^{13}\ \mathrm{cm}^{-3}$, $R = 2\,\mathrm{m}$, and $a = 50\ \mathrm{cm}$. If we choose a toroidal mode number of $n = 5$, and a wave frequency of 25 MHz ($= \frac{1}{2} f_{ci}$), then $\delta_1 \delta_2 = 5 \times 10^{-3}$. From Fig. 2, we then see that values of Q greater than 100 are possible.

Acknowledgment.

The series solutions (Case 1) were obtained using MACSYMA, a symbolic computation system at the Laboratory for Computer Science, Massachusetts Institute of Technology.

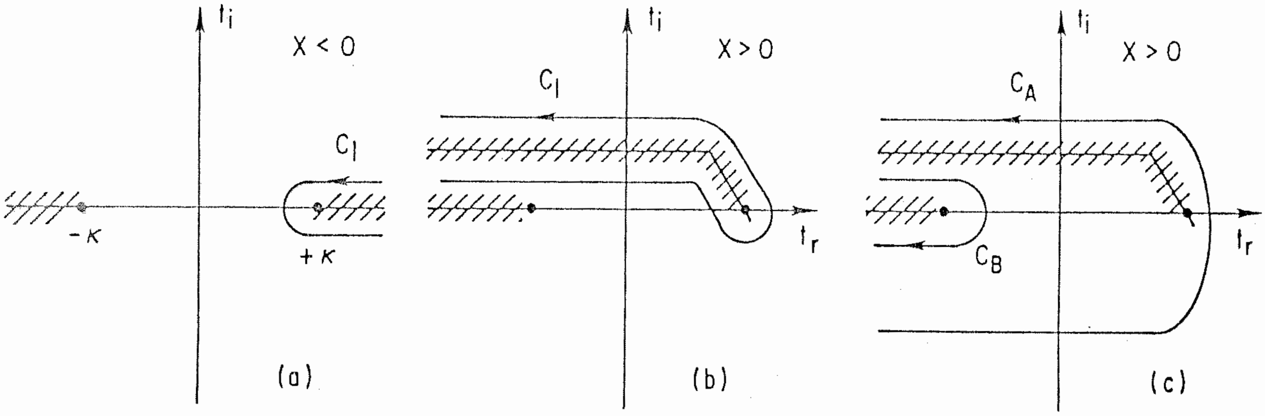

Fig. 1. The contours used in evaluating (9).

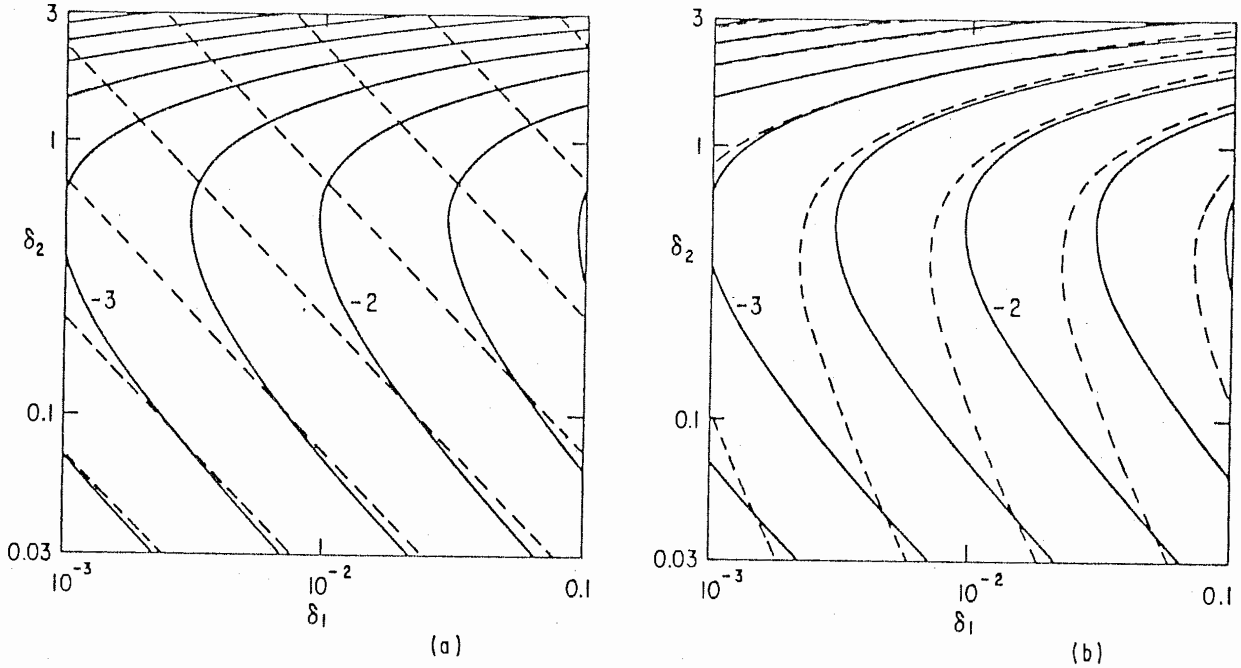

Fig. 2. Contours of $\log_{10}(q)$. The contours are equally spaced at 5 dB intervals. Solid lines give q as determined by integrating (5) numerically. Dashed lines show q as given by (a) (8) and (b) (11).