

## MULTIGRID SOLUTION OF THE FOKKER-PLANCK EQUATION

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*An application is described of the multigrid numerical method to solve the Fokker-Planck equation for a magnetized plasma.*

**Introduction.** The evolution of the distribution function of each species in a homogeneous plasma is governed by the Fokker-Planck equation,<sup>1</sup> which is a convection-diffusion equation in momentum space. Besides collisions the convection-diffusion formulation can represent also wave-plasma interaction and the force due to an electric field. Solution of this equation is of interest in magnetic confinement fusion research to study processes of heating, current-drive, radiation emission, and electron runaway. The present paper describes the numerical approach used in a new Fokker-Planck code. The principal feature is the use of the multigrid method,<sup>2</sup> which makes the code efficient both in terms of execution time and of memory requirements. From a physics point of view the code is intended to be useful for the range of problems indicated before.

**Mathematical formalism.** We consider the evolution of the distribution function  $f_a(\mathbf{u}, t)$  for species  $a$ , described by the equation

$$\frac{\partial f_a}{\partial t} + \frac{\partial}{\partial \mathbf{u}} \cdot \left( \mathbf{F}_a f_a - \mathbf{D}_a \cdot \frac{\partial f_a}{\partial \mathbf{u}} \right) = g_a. \quad (1)$$

The variable  $\mathbf{u}$  represents the ratio of momentum to species mass. The flux  $\mathbf{S}_a = \mathbf{F}_a f_a - \mathbf{D}_a \cdot \partial f_a / \partial \mathbf{u}$  is composed of three terms,  $\mathbf{S}_a = \mathbf{S}_a^c + \mathbf{S}_a^w + \mathbf{S}_a^e$ , associated with collisions, with wave-particle interaction, and with an electric field, respectively. (The coefficients  $\mathbf{D}_a$  and  $\mathbf{F}_a$  are decomposed in an analogous fashion.) The right hand side  $g_a$  is a source term of some parametric form. In the case of a magnetized plasma the distribution function may be assumed to be symmetric about the direction of the magnetic field, so that the problem involves only two velocity-space dimensions and time.

The collisional flux has the form  $\mathbf{S}_a^c = \sum_b \mathbf{S}_{ab}^c$  where the summation is over all species. According to Landau<sup>1</sup> the corresponding coefficients are

$$\mathbf{D}_{ab}(\mathbf{u}) = \frac{q_a^2 q_b^2}{8\pi\epsilon_0^2 m_a^2} \log \Lambda_{ab} \int \mathbf{U}(\mathbf{u}, \mathbf{u}') f_b(\mathbf{u}') d^3 \mathbf{u}', \quad (2a)$$

$$\mathbf{F}_{ab}(\mathbf{u}) = -\frac{q_a^2 q_b^2}{8\pi\epsilon_0^2 m_a m_b} \log \Lambda_{ab} \int \left( \frac{\partial}{\partial \mathbf{u}'} \cdot \mathbf{U}(\mathbf{u}, \mathbf{u}') \right) f_b(\mathbf{u}') d^3 \mathbf{u}'. \quad (2b)$$

Here,  $q_a$  and  $q_b$  are the species charge,  $m_a$  and  $m_b$  are the species mass,  $\epsilon_0$  is the vacuum dielectric permittivity, and  $\log \Lambda_{ab}$  is the Coulomb logarithm. The kernel

$\mathbf{U}$  is given by Landau as  $\mathbf{U} = s^{-3}(s^2\mathbf{I} - \mathbf{ss})$  in which  $\mathbf{s} = \mathbf{v} - \mathbf{v}'$  is the relative velocity between the colliding particles;  $\mathbf{v} = \mathbf{u}/\gamma$  with  $\gamma = (1 + u^2/c^2)^{1/2}$ , and likewise for  $\mathbf{v}'$ . Landau derived his expression under the assumption of Coulomb collisions, but using the correct relativistic particle dynamics. The fully relativistic collision operator is developed in Refs. 3 and 4. For the nonrelativistic case, it is well known<sup>5,6</sup> that the integrals appearing in Eq. (2) can be expressed in terms of the solution of a pair of differential equations. This potential formulation has recently been extended to the fully relativistic collision term.<sup>7</sup>

The flux  $\mathbf{S}_a^w$  due to wave-plasma interaction is obtained from quasilinear diffusion theory.<sup>8</sup> This must be generalized to include relativistic effects in the resonance condition and the diffusion paths. In many basic studies it is sufficient merely to use model forms (e.g. Ref. 9). The term  $\mathbf{S}_a^w$  is purely diffusive ( $\mathbf{F}_a^w = \mathbf{0}$ ) with an anisotropic diffusion tensor  $\mathbf{D}_a^w$ . The quasilinear diffusion coefficients may be discontinuous and may dominate over the collisional coefficients in some regions of momentum space.

The flux due to the electric field has the form  $\mathbf{S}_a^e = (q_a/m_a)\mathbf{E}f_a$ , and is therefore purely convective. In the electron runaway regime it dominates over the other terms.

The right hand side  $g_a$  is usually associated with real space transport. A nonzero right hand side is required to compensate for heating and runaway effects. We use a parameterized form for  $g_a$  and optimize the parameters to obtain a solution  $f_a$  having a specified density and temperature.

**Numerical implementation.** In recent years the multigrid method<sup>2,10</sup> has become increasingly popular for the solution of partial differential equations. Application to regular elliptic problems on simple domains is a routine exercise; an example in the context of plasma physics is the solution of the Grad-Schlüter-Shafranov equation.<sup>11</sup> Having seen many successful applications in fluid dynamics (e.g. Refs. 10 and 12) we decided to use multigrid for the solution of the Fokker-Planck equation. The special difficulties that one has to be aware of when developing a numerical procedure to solve Eq. (1) include: (1) the nonlinearity introduced through the collision operator; (2) the mixed nature of the equations (elliptic in the bulk and hyperbolic in the tail of the distribution, particularly for runaway problems); (3) the tensor character, strong anisotropy, and possible discontinuities in the diffusion coefficients; (4) the possible need to solve for several coupled species.

At the present stage of the work we have a code that solves Eq. (1) with characteristic multigrid efficiency, but the implementation of the physics that goes into the coefficients  $\mathbf{D}_a$  and  $\mathbf{F}_a$  is still incomplete. In particular we consider just one non-Maxwellian species, the collision operator is calculated only with respect to the isotropic part of the distribution function,<sup>13,14</sup> and model forms are used for the wave-plasma interaction. We do believe that we have addressed the main numerical complexities associated with the Fokker-Planck equation, and that further development of the physics in the code will not affect the general approach. This will now be described.

Equation (1) is discretized on each multigrid level by using the finite-volume method. The distribution function  $f_a$  is represented by its average values on the cells of the mesh, and the flux  $\mathbf{S}_a$  is represented by its average values through the cell faces. Other quantities are also represented by averaged values on cells or cell faces, as appropriate. This discretization was chosen because it makes it easy to satisfy discrete conservation equations. Given the local values of the coefficients  $\mathbf{F}_a$  and  $\mathbf{D}_a$  in Eq. (1) our code is written in a way that allows different meshes to be used, but at present the computation of the collision operator requires a spherical mesh (with rotational symmetry about the direction of the magnetic field). The mesh spacing may be nonuniform.

The derivatives in velocity space are discretized using an upwind-weighted scheme.<sup>15,16</sup> This method is formally second-order accurate, stable at all values of the ratio of convection to diffusion, and ensures the correct exponential behaviour of the solution at velocities large compared to the thermal velocity. The time discretization is fully implicit backward Euler. This method is only first-order accurate, but linearly stable at any timestep and therefore particularly suitable for steady state calculations.

At each iteration within an implicit timestep the discretized equations appear in nine-point molecule form, each cell being coupled to its eight nearest neighbours. For relaxation procedure we have employed alternating direction line relaxation and also incomplete L\*U (ILU) decomposition. Line relaxation allowed a very fast parallel implementation, but the ILU procedure turned out to be more effective per sweep. A reasonable balance between the work spent on calculating the coefficient matrix and the work spent on relaxation, and an optimum overall performance, was obtained by using around 10 iterations of line relaxation or 2 iterations of ILU for each evaluation of the coefficients, and for these parameters the ILU procedure provided the most rapid convergence (error reduction vs. CPU-time).

A clear advantage of ILU relaxation over point- or line-methods occurs in the regime of convection-dominated transport, in which case (for a suitable ordering of the unknowns) the matrix becomes triangular and is exactly factored by ILU. In practical calculations, where the direction of convection may vary over the domain or may not be known in advance, it is necessary to employ on alternate iterations two different orderings of the unknowns. This is done in our code. A second advantage of ILU is that it is robust in the presence of off-diagonal terms in the diffusion tensor, such as occur in the Fokker-Planck equation.

Our algorithm is of the full multigrid, full approximation storage variety.<sup>2</sup> Thus an approximate solution at a coarser level is used as a first guess on the next finer level, the equations are discretized in the same manner on all levels, and the solution is represented on all levels. Local defect extrapolation is employed to increase the accuracy of the computed solution. Within the multigrid sequence each cell of a coarser mesh is composed of  $2 \times 2$  cells from the next finer mesh. The coarsest mesh can consist of as few as  $2 \times 2$  cells. The coarse-to-fine interpolation of mesh functions relies on a piecewise linear representation within each cell of the coarse mesh, and the fine-to-coarse restriction is obtained by averaging four fine mesh

quantities, weighted by cell volume. These interpolation and restriction operators conserve particles.

We use semi-adaptive criteria for switching between multigrid levels. An interpolation, say from level  $l-1$  to level  $l$ , is made when the norm of the residual  $\|r_{l-1}\|$  is reduced below a certain tolerance  $\epsilon_{l-1}$ , where we set  $\epsilon_{l-1} = \max(c_0\epsilon_l, c_1\|r_l\|)$ , typically with  $c_0 = 0.2$  and  $c_1 = 0.5$  (a different criterion is used for the first transfer from level  $l-1$  to level  $l$ ). A restriction, say from level  $l$  to level  $l-1$ , is made when the convergence criterion on level  $l$  has not been met within a specified number of passes, typically one or two. (A pass refers to one evaluation of the coefficient matrix, and involves several iterations of the relaxation procedure.)

A standard test problem is an electron runaway calculation. An accurate converged solution is obtained in two to five multigrid cycles, involving four to ten passes over the finest mesh. This includes the work for the parameter optimization on the right hand side of Eq. (1), needed to obtain a steady solution. Computing times on a Cray-1 are in the range of 4-10 seconds for a  $64 \times 192$  mesh. We conclude that multigrid provides an efficient means of solving the Fokker-Planck equation for a magnetized plasma.

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